Some New Applications of Weyl's Multi-Polarization Operators

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Introduction: Part One

Let $\mathfrak{A}_N$ denote the enveloping algebra of $\mathfrak{gl}_N(\mathbb{C})$.

In Chapter Two of *The Classical Groups*,[W], Weyl constructed certain distinguished elements of $\mathfrak{A}_N$, which he called *quasi-compositions*; for a more modern presentation of this material, see (Fulton and Harris,[FH,pp 514-515].). One of the main purposes of the present paper, is to direct the attention of representation-theorists to these remarkable (but rather neglected) objects in $\mathfrak{A}_N$.

These objects will here be referred to, equivalently, as "*multi-polarizations*" or as "*Weyl polarizations*"; their definitions will be explained in §1.3 below. Weyl used them for the purpose of proving the celebrated Capelli identities in $\mathfrak{A}_N$, which in turn he utilized to prove what is often called (following Weyl's nomenclature) the 'First Fundamental Theorem of Invariant Theory' for the classical groups.

But, in fact, these Weyl polarizations are not necessary, (and are no longer usually used), for either of these two original purposes. Indeed, Capelli himself did not require (nor mention) these multi-polarizations, at least not in [Cap1](the reference cited by Weyl in [Weyl], p.39), nor in [Cap2] (as cited by Young in [Young], pp.64–71), nor in [Cap3]. (For interesting summaries of Capelli's work, cf. ([Young,loc.cit.], [Umeda].)

Let it be emphasized at this point, that the present paper is **NOT** further concerned, either with the Capelli identities, or with the First Fundamental Theorem of Invariant Theory.

Thus, to make plausible the claim (here proposed) that the Weyl polarizations deserve further investigation, would seem to require obtaining some newer applications of these objects, to topics of more current interest, and with results which have not been obtained by other methods. Two such applications will be presented in the following paper, one to the Verma-Shapovalov element (mapping one Verma module into another), the other

to the remarkable complexes (constructed by A.Zelevinsky in [Zel]), which (together with their 'conjugates', under the interchange of symmetrization and alternation), furnish higher syzygies, for the Weyl modules defined in [CL], and for the Plücker equations which define the dual Weyl modules (also called shape functors or Schur functors).

We postpone until Part Two of the introduction, a sketch of issues involved in the second of these new applications of the Weyl polarizations.

Here are the main results concerning the Weyl polarizations, to be established in the present paper:

I) Weyl's original construction of the Weyl polarizations, as differential operators, is recalled in Def.1.3.1 below. A remarkably simple and quite different-looking, purely combinatorial construction (which, it is hoped, is new) is given in Def.1.3.2; these two constructions are proved equivalent in Th.2.3.1. It is this combinatorial construction which plays a natural role in the study of the syzygies of the Plücker equations, as will be explained in §3.1, and which make it plausible (it is here proposed) that still further applications may exist for the Weyl polarizations.

II) The definition of the Verma-Shapovalov element is reviewed in §3.3 below. The work of Zelevinsky ([Zel]) and its later development by Akin ([Akin1,2]) relates this element to the above-mentioned complexes, and hence to the Weyl polarizations. (I should like to thank Bhama Srinivasan for calling this connection to my attention.) There is obtained below an explicit formula, stated in §3.3, expressing the action on

$$\underbrace{SV \otimes SV \otimes \cdots \otimes SV}_{N\ times}$$

of the Verma-Shapovalov element for $\mathfrak{sl}_N(\mathbb{C})$, as a $\mathbb{Z}$-linear combination of Weyl polarizations. (This appears not to be an obvious consequence of two explicit formulas for the Verma-Shapovalov element which have appeared in the literature, but which do not utilize the Weyl polarizations, viz. the formula given by Carter and Lusztig in [CL], and that given by[MFF].)

III) Explicit formulas in terms of the Weyl polarizations, which the author believes are new, are given in §3.1, for the differentials in the Zelevinsky complex discussed in the following sub-section.

These formulas are shown in §3.2 to check with earlier precise data for the special case $N = 3$, presented by Doty in ([Doty],p.134–136), and there attributed to Verma.

IV) The proofs of the results just cited, utilize formulas (analogous to the Pieri formulas) for the product of a Weyl polarization by an elementary polarization.These formulas are derived in §2.1 and §2.2.

The present author wishes to acknowledge his debt to D.-N. Verma, for explaining in a number of public lectures over the past 20 years or so, the problem of explicitly defining the differentials in the above-mentioned complex; and to acknowledge in addition, a number of very illuminating conversations with Verma concerning these matters. The author is also indebted to J.Humphreys and A.Zelevinsky, for helpful suggestions concerning the rather large literature involving the Shapovalov element.

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Introduction: Part Two The Zelevinsky Complex

The Weyl polarizations seem extremely well-suited, to furnishing explicit formulas for the differentials in certain remarkable complexes which occur in representation-theory.

The starting point here is the mid-nineteenth-century Jacobi-Trudi identity between two symmetric polynomials in $x_1, \cdots, x_m$, namely

$$s_\alpha = \det\,(h_{\alpha_i - i + j}) = \begin{vmatrix} h_{a_1} & h_{a_1+1} & \cdots & h_{a_1+N-1} \\ h_{a_2-1} & h_{a_2} & \cdots & h_{a_2+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{a_N-N+1} & h_{a_N-N+2} & \cdots & h_{a_N} \end{vmatrix} \tag{1}$$

Here

$$\alpha = (a_1 \geq a_2 \geq \cdots \geq a_N \geq 0)$$

denotes a partition (possibly terminated by a string of zeros), s_α denotes the associated Schur function, and $h_i = h_i(x_1, \cdots, x_m)$ has its usual meaning (i.e. the sum of all monomials in $x_1, \cdots, x_m$ of total degree i; in particular, this is 1 if $i = 0$, and 0 if $i < 0$).

Let us denote the Grothendieck ring of polynomial representation-functors of the group-scheme $GL_{\mathbb{C}}$ by $GL_{\mathbb{C}}{}^{\wedge}$ (Caution: The results in the present paper relevant to the Zelevinsky complex, are only asserted in characteristic 0—this is signalled by our choice of $\mathbb{C}$ as groundfield. On the other hand, many of the results obtained below concerning the Verma-Shapovalov element, are valid for arbitrary ground-ring. Finally, only the Lie algebra A_N is here studied.)

A fruitful heuristic principle in the representation-theory of $GL_{\mathbb{C}}$ and the symmetric groups, is that whereby identities involving symmetric polynomials can in many cases be re-interpreted as equations in the Grothendieck ring $GL_{\mathbb{C}}{}^{\wedge}$. Let us now examine the result of applying this principle to the equation (1) above:

In this heuristic manner, there is strongly suggested the existance of a complex in the Grothendieck ring $GL_{\mathbb{C}}{}^{\wedge}$, which is to be an exact sequence, such that equating to 0 its Euler-Poincare characteristic, yields precisely the Jacobi-Trudi identity (1). Over the last 25 years, a number of experts (discussed in further detail in §3.1) have considered the construction of such a complex. This project seems to have been initiated by A. Lascoux in [Las]; however, the complex constructed by Lascoux, unlike the later Zelevinsky complex, has differentials which are basis-dependent, (utilizing the combinatorics of Young tableaux) and so are not $GL(V)$-linear. These various attempts to construct such a complex, have produced the following mixed results:

The *terms* in these complexes may be precisely specified, in a simple fashion for which all authors seem in agreement; the precise definitions of the *differentials* in these complexes has been more problematical, and it is (in the present author's opinion) only the work of Zelevinsky in [Zel] which has first succeeded in producing mathematically acceptable constructions of these differentials as natural transformations. One of the main purposes of the present paper, is that of rendering more transparent the construction of these differentials, utilizing properties (to be established in Chapter Two below) of Weyl polarizations. This work logically divides into two parts:

1) We shall construct *explicitly* linear transformations which play the role of the differentials in these complexes. This will be done in §3.1 below, and this construction uses nothing of the machinery of Verma modules and the homomorphisms between these. These results appear to be quite new (except for the very special explicit results of Verma for the case $N = 3$ of 3-part partitions, explained in Doty [Doty,pp.134–136], which agree with the present more general results—as will be verified below in §3.2).

2)In Chapter Four, the explicit maps furnished by this 'elementary' construction will be proved to coincide precisely, with the maps provided by Zelevinsky in terms of the theories of Verma[Verma1,2], Bernstein-Gel'fand-Gel'fand[BGG] and Shapovalov[Shap]. To the present author's best knowledge, it is only with the work of Zelevinsky that such a comparison becomes possible— there was in this sense no *specific* complex in $GL_{\mathbb{C}}{}^{\wedge}$ in existence earlier, for which such comparison could make sense. It is for this reason that it seems appropriate to use here the terminology 'Zelevinsky complex'. Some important later work by Akin, Doty and Maliakas, developing these ideas of Zelevinky, may be found in [Akin1,2], [Doty2] and [Mal]. In particular, the present paper is heavily indebted to the observation of Akin (in [Akin2,p. 418]) relating the differentials in the Zelevinsky complex to the Verma-Shapovalov elements—this will be discussed further in §3.3 below.

In the present author's opinion, there should exist a more direct combinatorial proof that the complex constructed in §3.1 is exact. At present, the only proof in the author's possession, involves the detour (presented in Chapter Four below) through the theory of Verma modules and the work of Zelevinsky. One benefit of this detour is the explicit formula, presented in §3.3 below, expressing (as mentioned earlier) the action on $S^{\otimes N}$ of each Verma-Shapovalov element for A_N as a $\mathbb{Z}$-linear combination of Weyl polarizations.

Chapter 1 Weyl's Multi-Polarization Operators

§1.1 **Notation**

Throughout the following paper, the ground-field will be $\mathbb{C}$. $\bigotimes$ will always denote $\bigotimes_{\mathbb{C}}$. Let N be a positive integer.We denote by $\underline{N}$ the set $\{1, ..., N\}$ and by $S^{\otimes N}$ the functor (on the category of complex vector-spaces and $\mathbb{C}$-linear transformations) which assigns to each complex vector-space V, the complex vector-space

$$S^{\otimes N}V = \underbrace{SV \otimes SV \otimes \cdots \otimes SV}_{\text{N factors}}$$

(where SV is the usual symmetric algebra on V). Similarly, we define $\Lambda^{\otimes N}$ to be the functor given by

$$\Lambda^{\otimes N}V = \underbrace{\Lambda V \otimes \Lambda V \otimes \cdots \otimes \Lambda V}_{\text{N factors}}$$

For $i, j \in \underline{N}$, we denote as usual by $E_{i,j}$ the $N \times N$ matrix whose only non-zero entry is a 1 in row i and column j. We denote by $\mathfrak{A}_N$ the enveloping algebra of $\mathfrak{gl}(N, \mathbb{C})$, and

by $(\mathfrak{A}_N)^0$ the enveloping algebra of $\mathfrak{sl}(N,\mathbb{C})$ (considered as a subalgebra of $\mathfrak{A}_N$). In all computations in the present paper, permutations will act on the *left* of elements, and linear transformations will act on the *left* of (column) vectors; thus compositions of linear transformations and of permutations are to be read right-to-left.

§1.2 Elementary Polarization Operators

If i and j are integers between 1 and N, there is defined a natural transformation

$$D_{i,j} : S^{\otimes N}V \to S^{\otimes N}V \tag{1.2.1}$$

(the *elementary polarization operator*), as follows. (We review this well-known concept in some detail, in preparation for its generalization in the next §.)

We recall two definitions for these $D_{i,j}$, the first only meaningful if V is finite-dimensional over a ground-field of characteristic 0, the second valid for an arbitrary module V over an arbitrary commutative ground-ring — and with both definitions equivalent for finite-dimensional V over a field of characteristic 0.

Suppose first that V is a finite-dimensional complex vector-space, with $\mathbb{C}$-basis

$$\mathcal{B} = (x_1, \ldots, x_M)$$

$\mathcal{B}$ induces $\mathbb{C}$-algebra isomorphisms

$$\phi_{\mathcal{B}} : SV \simeq \mathbb{C}[X_1, \ldots, X_M] \tag{1.2.2}$$

and

$$\Phi_{\mathcal{B}} : S^{\otimes N}V \simeq \mathbb{C}[X_1^{(1)}, \ldots, X_M^{(1)}; \ldots; X_1^{(N)}, \ldots, X_M^{(N)}] \tag{1.2.2a}$$

(where, in (1.2.2a), $X_1^{(1)}, \ldots, X_M^{(N)}$ denote MN independent indeterminates over $\mathbf{C}$). These maps will usually be treated implicitly as identifications. With the identification (1.2.2a) we then define, for $a_1, \ldots, a_N$ any natural numbers, the restriction of $D_{i,j}$ to $S^{a_1} \otimes \ldots \otimes S^{a_N}$ to be given by

$$D_{i,j} = \sum_{k=1}^{M} X_k^{(i)} \frac{\partial}{\partial X_k^{(j)}} : S^{a_1} \otimes \ldots \otimes S^{a_N} \to S^{a'_1} \otimes \ldots \otimes S^{a'_N} \tag{1.2.3}$$

where we have set

$$a'_l = \begin{cases} a_i + 1, & \text{if } l = i \\ a_j - 1, & \text{if } l = j \\ a_l & \text{if } l \neq i \text{ or } j \ . \end{cases}$$

(We set $D_{ij}|S^{a_1} \otimes \cdots \otimes S^{a_N} = 0$ if $a_j = 0$.) Note that while the operations $X_k^{(i)}$ and $\frac{\partial}{\partial X_k^{(j)}}$ on $S^{\otimes N}$ of course depend on $\mathcal{B}$, the combination (1.2.3) is readily verified to be independent of the choice of basis $\mathcal{B}$. Here is a basis-free equivalent definition, which does not require that V be finite-dimensional:

Let V be an arbitrary module over an arbitrary commutative ring R.

Suppose first that $i < j$; then the natural transformation $D_{i,j}$ may be equivalently defined (combinatorially rather than in the explicit form of a differential operator) as the operator

$$D_{i,j} : S^{a_1} \otimes_R \cdots \otimes_R S^{a_N} \to S^{a'_1} \otimes_R \cdots \otimes_R S^{a'_N}$$

which maps

$$\sigma^{(1)} \otimes \cdots \otimes \sigma^{(N)} = (v_1^{(1)} \cdot \ldots \cdot v_{a_1}^{(1)}) \otimes (v_1^{(2)} \cdot \ldots \cdot v_{a_2}^{(2)}) \otimes (v_1^{(N)} \cdot \ldots \cdot v_{a_N}^{(N)}) \tag{1.2.4}$$

(with all $v_q^{(p)} \in V$) into the element

$$\sum_{\lambda=1}^{a_j} \sigma^{(1)} \otimes \cdots \otimes (v_\lambda^{(j)} \cdot \sigma^{(i)}) \otimes \cdots \otimes (v_1^{(j)} \cdot \ldots \widehat{v_\lambda^{(j)}} \cdot \ldots \cdot v_{a_j}^{(j)}) \otimes \cdots \otimes \sigma^{(N)}$$

with a precisely similar definition (except that the order of the two main parentheses is reversed) if $i > j$—while if $i = j$, $D_{i,i}$ acts on $S^{a_1} \otimes \cdots \otimes S^{a_N}$ as multiplication by a_i .

In all three cases, $D_{i,j}$ operates on an element

$$\omega = (v_1^{(1)} \cdot \ldots \cdot v_{a_1}^{(1)}) \otimes \cdots \otimes (v_1^{(N)} \cdot \ldots \cdot v_{a_N}^{(N)})$$

like this:

In all possible ways remove a factor $v_\lambda^{(j)}$, where $1 \le \lambda \le a_j$, from the j-th tensorand $\sigma^{(j)}$ in ω, and re-insert this $v_\lambda^{(j)}$ into the i-th tensorand; then add all the a_j results thus obtained.

EXAMPLE:

$D_{13}(x_1x_2x_3 \otimes y_1 \otimes z_1z_2) = x_1x_2x_3z_1 \otimes y_1 \otimes z_2 + x_1x_2x_3z_2 \otimes y_1 \otimes z_1$
and

$D_{11}(x_1x_2x_3 \otimes y_1 \otimes z_1z_2) = 3x_1x_2x_3 \otimes y_1 \otimes z_1z_2$

(Note: Some authors prefer to say the same thing in still a third more high-falutin' way, by expressing these elementary polarizations (in the obvious way) in terms of the

component

$$S^aV \to S^{a-1}V \otimes V,\, x_1 \otimes \cdots \otimes x_a \mapsto \sum_{i=1}^{a}(x_1 \otimes \cdots \hat{x}_i \cdots x_a) \otimes x_i$$

of the comultiplication map of the Hopf algebra SV.)

We assume,for the rest of this section, that the ground-ring is $\mathbb{C}$. Using any of the preceding equivalent definitions for $D_{i,j}$, it is readily verified that these elementary polarization operators on $S^{\otimes N}V$ obey the commutation relations

$$[D_{ij}, D_{kl}] = \delta_{jk}D_{il} - \delta_{il}D_{kj}$$

which imply the existence of an action P of $\mathfrak{A}_N$ on $S^{\otimes N}$ by natural transformations, uniquely specified by

$$P(E_{ij}) = D_{ij}\,.$$

In fact, we thus obtain, it is well-known, a natural Lie-algebra monomorphism (here to be referred to as the **Capelli injection**)

$$C : \mathfrak{A}_N \to \text{Nat Tsf}(S^{\otimes N}, S^{\otimes N}), E_{i,j} \mapsto D_{i,j}\,. \tag{1.2.5}$$

We shall, for the remainder of this paper, treat the Capelli injection as an identification. **In particular, this paper will always identify $D_{i,j} = P(E_{i,j})$ (the natural transformation on $S^{\otimes N}$) with $E_{i,j} \in \mathfrak{A}_N$.** (Usually, the notation $E_{i,j}$ will be used for both.)

In the next sub-section, the Weyl multi-polarization will be defined by specifying its image (via C) as a natural endomorphism of $S^{\otimes N}$.

CAUTION: For fixed V, the action of $\mathfrak{A}_N$ on $S^{\otimes N}V$ by $\mathbb{C}$-linear transformations, need *not* be faithful; it is for this reason that we shall instead use the faithful action by natural endomorphisms of the functor $S^{\otimes N}$ (i.e., work in terms of 'generic' V.)

§1.3 Multi-polarization Operators on $S^{\otimes N}$

We now turn to certain somewhat intricate operations on $S^{\otimes N}$, for whose construction the earliest source known to the present author is H.Weyl (cf.[Weyl,p.39]).

Let $\Pi^N_\pm$) denote the free Abelian group (written additively) on the set of N^2 matrices $E_{i,j}$ explained in §1.1. Thus the elements in $\Pi^N_\pm$ are of the form

$$\sigma = \sum_{i,j \in \underline{N}} \sigma_{i,j} E_{i,j} \qquad (\text{all } \sigma_{i,j} \in \mathbb{Z}); \tag{1.3.1}$$

each such σ may also be regarded as an $N \times N$ matrix $||\sigma_{i,j}||$ with integer entries.

Those σ for which all $\sigma_{i,j} \geq 0$ will be called **N-shifts**; the set of N-shifts will be denoted by Π^N. (Thus, Π^N is the free Abelian semi-group on the set of $E_{i,j}$'s.) An element σ in $\Pi^N_\pm$ will be called **effective** if all $\sigma_{i,j}$ are non-negative (i.e., if σ is an N-shift) and **non-effective** otherwise.

In [Weyl, loc.cit.], Weyl associates (in a slightly different notation) to every N-shift $\sigma \in \Pi^N$, a transformation

$$P(\sigma) : S^{\otimes N} V \to S^{\otimes N} V$$

which is now to be defined, and which we shall call the **Weyl polarization operator associated to** σ**;** we shall also sometimes refer to these as **multi-polarization operators.** They include, as a special case, the elementary polarization operators discussed in the preceding section.

It will be convenient, for the N-shift given by (1.3.1), to set

$$\sigma! = \prod_{i,j \in \underline{N}} (\sigma_{i,j}!) \tag{1.3.2}$$

(and to define $\sigma!$ to be 0 if σ is non-effective.)

Again, (as in §1.2) we shall give two definitions (equivalent where both are defined). The first, that given in [Weyl, loc.cit.]; cf. also (Fulton and Harris[FH, pp.514–515]) requires that V be a finite-dimensional complex vector-space,and involves a choice of basis for V.The second is defined for V an arbitrary module over an arbitrary commutative ring.

DEFINITION 1.3.1

Consider first the case that V is a finite-dimensional complex vectorspace, with $\mathbb{C}$-basis

$$\mathcal{B} = (x_1, \ldots, x_M)$$

(so M =dim V). As before, we use $\mathcal{B}$ to define the endomorphisms (1.2.2) and (1.2.2a). Now write (in the commutative semi-group Π^N)

$$\sigma = E_{i_1,j_1} + E_{i_2,j_2} + \cdots + E_{i_L,j_L}.$$

Then we define the **non-normalized Weyl polarization**

$$P_0(\sigma) : S^{\otimes N}V \to S^{\otimes N}V$$

to be the endomorphism of $S^{\otimes N}V$ given by:

$$P_0(\sigma) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{M} \cdots \sum_{k_L=1}^{M} X_{k_1}^{(i_1)} X_{k_2}^{(i_2)} \cdots X_{k_L}^{(i_L)} \frac{\partial}{\partial X_{k_1}^{(j_1)}} \frac{\partial}{\partial X_{k_2}^{(j_2)}} \cdots \frac{\partial}{\partial X_{k_L}^{(j_L)}} \tag{1.3.3}$$

There is (as will become apparent) some advantage to considering instead the **normalized Weyl polarizations**

$$P(\sigma) \stackrel{\text{def}}{=} (\frac{1}{\sigma!})P_0(\sigma) \tag{1.3.4}$$

(In particular, it is with this normalization that Th.2.3.1 below becomes valid.)

When mention is made below simply of Weyl polarizations, these normalized polarizations $P(\sigma)$ are always to be understood. Although $P_0(\sigma)$ and $P(\sigma)$ are basis-independent and natural in V, these facts are perhaps not obvious at this stage (but will be immediate corollaries of Th. 2.3.1 below).

REMARKS ON NOTATION: Weyl in [Weyl, loc.cit.] uses the notation

$$\Delta_{i_1,j_1}\Delta_{i_2,j_2}\cdots\Delta_{i_N,j_N}$$

for the expression (1.3.3), which he calls a **quasi-composition**, remarking that (unlike the ordinary composition $D_{i_1,j_1}\cdots D_{i_N,j_N}$) it is unchanged by rearrangement of the factors. It seems perhaps more lucid to use instead a notation such as $P_0(\sigma)$ for (1.3.3), particularly since it is *not* always true that

$$P_0(E_{i_1,j_1} + E_{i_2,j_2})(= \Delta_{i_1,j_1}\Delta_{i_2,j_2} \text{ in Weyl's notation})$$

is the composite of $D_{i_1,j_1} = P_0(E_{i_1,j_1})$ and $D_{i_2,j_2} = P_0(E_{i_2,j_2})$.

Weyl's original notation (which was elegantly suited to his proof in [Weyl,Chapter II, §4] of the Capelli identity) seems in the spirit of the 'umbral' notation of nineteenth century invariant theory, in which 'symbolical products' occur, i.e. objects which look like products but are not, and require re-interpretation.

Also, the additive notation we have adopted in Π^N, has the advantage of avoiding confusion between e.g. the N-shift

$$E_{1,2} + E_{2,3} = E_{2,3} + E_{1,2} \in \Pi^N$$

and the elements

$$E_{1,2}E_{2,3} \neq E_{2,3}E_{1,2}$$

in the enveloping algebra $\mathfrak{A}_N$.

By the *weight vector* $wt(\sigma)$ of an N-shift σ, will be meant the N-tuple

$$wt(\sigma) = (wt_1(\sigma, \dots, wt_N(\sigma) \tag{1.3.5}$$

where

$$\begin{aligned} wt_i(\sigma) &= \sum_{j=1}^{N} \sigma_{i,j} - \sum_{j=1}^{N} \sigma_{j,i} \qquad\qquad (1.3.5a) \\ &= (i^{th} \text{ row-sum of } \sigma, \text{ minus the } i^{th} \text{ column-sum }) \end{aligned}$$

For $a_1, a_2, \cdots, a_N$ any natural numbers, it is readily verified that $P(\sigma)$ maps

$$S^{a_1} \otimes \cdots \otimes S^{a_N} \to S^{a_1 + wt_1(\sigma)} \otimes \cdots \otimes S^{a_N + wt_N(\sigma)} \tag{1.3.6}$$

(This is 0 if any $a_i + wt_i(\sigma) = b_i$ is negative, in accordance with the usual convention that $S^b V = 0$ if $b < 0$.)

Note that $\sum_{i=1}^{N} wt_i(\sigma) = 0$.

Now let us drop the assumption that V is finite-dimensional. Our next goal is to give a second definition of $P(\sigma)$ using combinatorial concepts rather than partial derivatives (but agreeing, as will be proved below, with Def. 1.3.1 where the latter has been defined.)

Definition 1.3.2. *If V is a module over the commutative ring R, and if*

$$\alpha = (a_1, \dots, a_N)$$

is any N-tuple of non-negative integers, then in order to define the restriction

$$P(\sigma, \alpha) := P(\sigma)|(S^{a_1}V \otimes \cdots \otimes S^{a_N}V)$$

its action is to be given on the generating elements

$$\omega = (x_1^{(1)} \cdot x_2^{(1)} \cdot \ldots \cdot x_{a_1}^{(1)}) \otimes \cdots \otimes (x_1^{(N)} \cdot x_2^{(N)} \cdot \ldots \cdot x_{a_N}^{(N)})$$

(with all $x_p^{(q)}$ elements of V), by the following rule:
Namely: $P(\sigma,\alpha)$ acts on ω by moving (in all possible ways) an **unordered** *collection of $\sigma_{i,j}$ letters x from the j-th tensor factor of ω to the i-th, for all i and j between 1 and N — NO LETTER BEING MOVED TWICE. The results are then summed, to obtain $P(\sigma)\omega$.*

(For the time being, we shall refer to the endomorphisms given by Definition 1.3.1 as **differential polarizations**, and those given by Definition 1.3.2 as **combinatorial polarizations**.This distinction is only a temporary one, since in Section 2.3 these two definitions will be proved equivalent where both are defined. Also, the preceding somewhat breezy version of Definition 1.3.2 will be restated more formally in Section 1.4.)

Note that (1.3.6) is clearly still the type of $P(\sigma)$, if $P(\sigma)$ is interpreted in the sense of Def.1.3.2.

Perhaps it will be helpful at this point to give some illustrative examples for Definition 1.3.2:

EXAMPLE 1.3.1

If

$$\sigma_1 = E_{1,2} + 2E_{13} + 3E_{32} \in \Pi^3,$$

i.e. if

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix},$$

then σ_1 has for weight-vector

$$wt(\sigma_1) = (3-0, 0-4, 3-2) = (3, -4, 1)$$

and the combinatorial polarization

$$P(\sigma_1, (a_1, a_2, a_3)) : S^{a_1}V \otimes S^{a_2}V \otimes S^{a_N}V \to S^{a_1+3} \otimes S^{a_2-4} \otimes S^{a_3+1}$$

is, (in accordance with Definition 1.3.2), the map which takes

$$\omega = (x_1 \cdot \ldots \cdot x_{a_1}) \otimes (y_1 \cdot \ldots \cdot y_{a_2}) \otimes (z_1 \cdot \ldots \cdot z_{a_3})$$

(all x's, y's and z's in V) into the sum of all $a_2 \cdot \binom{a_3}{2} \cdot \binom{a_2-1}{3}$ terms obtained by moving:

one letter y from the second tensorand into the first (corresponding to the entry $(\sigma_1)_{1,2} = 1$), two letters z from the third tensorand into the first (corresponding to $(\sigma_1)_{1,3} = 2$), and three letters y from the second tensorand to the third(corresponding to $(\sigma_1)_{3,2} = 3$), no letter being moved twice.

In other words, $P(\sigma_1)\omega$ is to equal the sum

$$\sum \left[(y_{j_1} z_{k_1} z_{k_2} x_1 \cdots x_{a_1}) \otimes (\frac{y_1 \cdots y_{a_2}}{y_{j_1} y_{J_1} y_{J_2} y_{J_3}}) \otimes (y_{J_1} y_{J_2} y_{J_3} z_1 \cdots \widehat{z_{k_1}} \cdots \widehat{z_{k_2}} \cdots z_{a_3}) \right],$$

extended over the indexing set indicated by

$$\sum_{1 \leq j_1 \leq a_2} \sum_{\substack{1 \leq J_1 < J_2 < J_3 \leq a_2 \\ j_1 \notin \{J_1, J_2, J_3\}}} \sum_{1 \leq k_1 < k_2 \leq a_3}$$

REMARKS:

A)We employ in this formula, (for ease of notation), two quite equivalent methods for denoting the deletion of factors from a product: in the second tensorand deletion of $y_{j_1} y_{J_1} y_{J_2} y_{J_3}$ is indicated as a division, while in the third tensorand, deletion of $z_{k_1} z_{k_2}$ is indicated by the usual "$\wedge$" notation. We could just as well have written these the other way around.

B)If $a_2 < 4$ then this sum is empty, and in accordance with the usual conventions for an empty sum, $P(\sigma_1)\omega = 0$.

EXAMPLE 1.3.2

If

$$\sigma_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \alpha = (a_1, a_2)$$

then $wt(\sigma_2) = (0, 0)$,and

$$P(\sigma_2, \alpha) : S^{a_1} V \otimes S^{a_2} V \to S^{a_1} V \otimes S^{a_2}$$

is the map which sends

$$\omega = (x_1 \cdot \ldots \cdot x_{a_1}) \otimes (y_1 \cdot \ldots \cdot y_{a_2})$$

into the sum

$$\sum_{1 \leq I_1 < I_2 \leq a_1} \sum_{\substack{1 \leq i \leq a_1 \\ i \notin \{I_1, I_2\}}} \sum_{1 \leq j \leq a_2} (y_j x_1 \cdots \widehat{x_i} \cdots x_{a_1}) \otimes (x_i y_1 \cdots \widehat{y_j} \cdots y_{a_2})$$

Note the effect of the diagonal entry $(\sigma_2)_{11} = 2$ in σ_2, is to select (in all possible ways), an unordered pair of letters $\{x_{I_1}, x_{I_2}\}$ in the first tensorand (both I_1 and I_2 being distinct from i_1), whereupon we proceed to leave these two letters untouched. (We may as well suppose $I_1 < I_2$.) Thus $P(\sigma_2)$ differs simply by the scalar factor $\binom{a_1-1}{2}$ from $P(\sigma_3)$ where

$$\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

—i.e.,

$$P(\sigma_2, \alpha) = \binom{a_1 - 1}{2} P(\sigma_3, \alpha)$$

where

$$P(\sigma_3, \alpha)\omega = \sum_{1 \le i \le a_1} \sum_{1 \le j \le a_2} (y_j x_1 \cdots \widehat{x_i} \cdots x_{a_1}) \otimes (x_i y_1 \cdots \widehat{y_j} \cdots y_{a_2}) \ .$$

§1.4 σ-selections

We next need to restate Definition 1.3.2 somewhat more formally. For this purpose, and also for later computations, the following notations will be convenient:

Let R be a fixed commutative ground-ring, and let V be an arbitrary R-module. Recall from §1.1 the notation

$$\underline{N} = \{1, ..., N\}$$

For any map

$$x : \underline{a} \to V$$

we define

$$x[[\,\underline{a}\,]] \stackrel{\text{def}}{=} x(1) \cdot x(2) \cdot \ldots \cdot x(a) \in S^a V$$

Note that such elements span $S^a V$ over R; and that similarly (for all non-negative integers $a_1, \cdots, a_N$)

$$S^{a_1} V \otimes \cdots \otimes S^{a_N} V$$

is spanned over R by the set of all elements of the form

$$\begin{aligned} \omega &= x^{(1)}[[\,\underline{a_1}\,]] \otimes \cdots \otimes x^{(N)}[[\,\underline{a_N}\,]] \\ &= (x^{(1)}(1) \cdot \ldots \cdot x^{(1)}(a_1)) \otimes \cdots \otimes (x^{(N)}(1) \cdot \ldots \cdot x^{(N)}(a_N)) \end{aligned} \tag{1.4.1}$$

(where, for $1 \le i \le N$, $x^{(i)}$ is an arbitrary map $\underline{a}_i \to V$).

More generally, for any sub-set E of $\underline{a}$, say

$$E = \{e_1, \cdots, e_L\} \subseteq \underline{a}\,, \qquad \text{with } e_1, \cdots, e_L \text{ distinct,}$$

and any map

$$x : \underline{a} \to V\,,$$

we define

$$x[[E]] \stackrel{\text{def}}{=} x(e_1) \cdot x(e_2) \cdot \ldots \cdot x(e_L) \in S^L V$$

Definition 1.4.1. *Let*

$$\sigma \in \Pi^N, \alpha = (a_1, a_2, \cdots, a_N) \in \mathbb{N}^N\,;$$

then by a σ-**selection from** α *will be meant a collection*

$$\{C_{i,j} : i \text{ and } j \in \underline{N}\}$$

such that, (for all i between 1 and N), $C_{1,i}, C_{2,i}, \ldots, C_{N,i}$ are pairwise disjoint subsets of $\underline{a}_i$, and such that (for all i and j between 1 and N), $\sigma_{i,j}$ is the cardinality of $C_{i,j}$. We denote by

$$\binom{\alpha}{\sigma}$$

the set of all such.

NOTE: Given such a σ-selection

$$\iota = \{C^\iota_{i,j} : i, j \in \underline{N}\} \in \binom{\alpha}{\sigma} \tag{1.4.2}$$

it will be convenient to set, for $1 \le i \le N$,

$$C^\iota_{0,i} \stackrel{\text{def}}{=} \{1, \ldots, a_i\} - (\bigcup_{j=1}^{N} C^\iota_{j,i}) \tag{1.4.3}$$

and

$$\sigma_{0,i} \stackrel{\text{def}}{=} a_i - \sum_{j=1}^{N} \sigma_{j,i} \tag{1.4.4}$$

Note that thus each $\underline{a}_i$ is the disjoint union of

$$C^\iota_{0,i}\,, C^\iota_{1,i}\,, \cdots, C^\iota_{N,i}\,,$$

and that

$$\#(C^{\iota}_{0,i}) = \sigma_{0,i}$$

The cardinality of

$$\binom{\alpha}{\sigma}$$

is then given by

$$\#\binom{\alpha}{\sigma} = \prod_{i=1}^{N} \frac{a_i!}{\sigma_{1,i}! \cdot \ldots \cdot \sigma_{N,i}! \cdot (a_i - \sigma_{1,i} - \cdots - \sigma_{N,i})!}$$

Let us now express $P(\sigma)\omega$ (where ω is given by eqn.(1.4.1)), in terms of the notation just explained. Recall that Def.1.3.2 spoke of "moving (in all possible ways) $\cdots \sigma_{i,j}$ letters x from the j-th tensor factor of ω to the i-th, for all i and j between 1 and N — NO LETTER BEING MOVED TWICE. The results are then summed, to obtain $P(\sigma)\omega$."

Clearly, such a 'possible way' of moving the letters of ω, is furnished precisely by a σ-selection

$$\iota = C^{\iota}_{i,j} \in \binom{\alpha}{\sigma}$$

which instructs us to move the $\sigma_{i,j}$ letters

$$x^{(j)}(e) \text{ where } e \in C^{\iota}_{i,j}$$

from the j-th tensor factor of ω to the i-th, thus obtaining a result which we shall denote by ω^{ι}, and whose precise value is given by:

$$\omega^{\iota} \stackrel{\text{def}}{=} (\omega^{\iota})_1 \otimes \cdots \otimes (\omega^{\iota})_N \tag{1.4.5}$$

where (for all i between 1 and N), we set

$$(\omega^{\iota})_i \stackrel{\text{def}}{=} (\prod_{j=1}^{N} x^{(j)}[[C^{\iota}_{i,j}]]) \cdot x^{(i)}[[C^{\iota}_{0,i}]] \in S^{a_i + wt_i(\sigma)} V \tag{1.4.6}$$

(cf. eqn.(1.3.5a) in §1.3). With the notation thus defined, the following equation may then be regarded as the deluxe version of Definition (1.3.2):

$$P(\sigma)\omega \stackrel{\text{def}}{=} \sum_{\iota \in \binom{\alpha}{\sigma}} \omega^{\iota} \, . \tag{1.4.7}$$

NOTE: Our formulas sometimes contain expressions of the form $P(\sigma')$ where σ' is an $N \times N$ matrix over $\mathbb{Z}$, not all of whose entries are ≥ 0. (In the terminology introduced in the beginning of §1.3, σ' is a non-effective element of $\Pi^N_\pm$)

To avoid any possible ambiguity, let us agree always to set, in such a case,

$$P(\sigma') = 0 \text{ if any } \sigma'_{i,j} < 0 \tag{1.4.8}$$

Section 2 The Product of a Multi-polarization by an Elementary Polarization

§2.1 The Recurrence for Combinatorial Polarizations

NOTE:Throughout this subsection, all multi-polarizations $P(\sigma)$ considered are to be understood in the sense of Definition 1.3.2 of the preceding §1.3. (We shall consider in the following subsection 2.2, multi-polarizations in the sense of Definition 1.3.1, i.e. in the sense of Weyl's original definition.)

Theorem 2.1.1. *Let $\sigma \in \Pi^N$ and let i,j be distinct integers between 1 and N.Then*

$$E_{i,j}P(\sigma) = (\sigma_{i,j}+1)P(E_{i,j}+\sigma) + \sum_{k=1}^{N}(\sigma_{i,k}+1)P(\sigma + E_{i,k} - E_{j,k}) \tag{2.1.1}$$

PROOF: By an obvious symmetry of the situation under study (namely, via the action of the symmetric group on N letters, upon the tensor product of N copies of SE), it is evident that it suffices to prove (2.1.1) in the special case $i = 2, j = 1$.

Thus, it suffices to verify that

$$E_{2,1}P(\sigma)\omega = (\sigma_{2,1}+1)P(E_{2,1}+\sigma)\omega + \sum_{k=1}^{N}(\sigma_{2,k}+1)P(\sigma + E_{2,k} - E_{1,k})\omega \tag{2.1.2}$$

for every generating element ω of the form (1.4.1). Let us then consider (utilizing the notation explained in §1.4)

$$E_{2,1}P(\sigma)\omega = \sum_{\iota \in \binom{\alpha}{\sigma}} E_{2,1}\omega^\iota = \sum_{\iota \in \binom{\alpha}{\sigma}} E_{2,1}[(\omega^\iota)_1 \otimes (\omega^\iota)_2 \otimes \cdots \otimes (\omega^\iota)_N]$$

$E_{2,1}\omega^\iota$ is the sum of all terms obtained by removing one letter u from the first tensorand

$$\begin{aligned}(\omega^\iota)_1 &= x^{(1)}[[\,\underline{a_1} - C^\iota_{2,1} - \cdots - C^\iota_{N,1}]] \cdot x^{(2)}[[C^\iota_{1,2}]] \cdot \ldots \cdot x^{(N)}[[C^\iota_{1,N}]] \\ &= x^{(1)}[[\,\underline{a_1} - C^\iota_{1,1} - \cdots - C^\iota_{N,1}]] \cdot \prod_{k=1}^{N} x^{(k)}[[C^\iota_{1,k}]]\end{aligned} \tag{2.1.3}$$

of ω^ι, and inserting it instead into the second tensorand

$$(\omega^\iota)_2 = x^{(1)}[[C^\iota_{2,1}]] \cdot x^{(2)}[[\,\underline{a_2} - C^\iota_{1,2} - C^\iota_{3,2} - \cdots - C^\iota_{N,2}]] \cdot \prod_{k=3}^{N} x^{(k)}[[C^\iota_{2,k}]] \qquad (2.1.3a)$$

—let us denote by $T(\omega^\iota, u)$ the term thus obtained. That is to say, for each letter u in (2.1.3) we set

$$T(\omega^\iota, u) = (\frac{(\omega^\iota)_1}{u}) \otimes (u \cdot (\omega^\iota)_2) \otimes (\omega^\iota)_3 \cdots \otimes (\omega^\iota)_N \qquad (2.1.4)$$

and we then have

$$E_{2,1}P(\sigma)\omega = \sum_{\iota \in \binom{\alpha}{\sigma}} \sum_{u} T(\omega^\iota, u) \qquad (2.1.5)$$

(the inner sum being extended over all letters u in (2.1.3).)

The next step is to decompose (2.1.5) into $N+1$ sub-sums, (according to where u comes from), as follows:

If $1 \le k \le N$, let us define $\mathcal{S}(k)$ to be the sum of those terms in (2.1.5) for which $u = x^{(k)}(e)$ with $e \in C^\iota_{1,k}$, i.e. we set

$$\mathcal{S}(k) \overset{\text{def}}{=} \sum_{\iota \in \binom{\alpha}{\sigma}} \sum_{e^{(k)} \in C^\iota_{1,k}} T(\omega^\iota, x^{(k)}(e^{(k)})) \qquad (2.1.6a)$$

(Here we adopt, as always, the convention which sets empty sums equal to 0— thus $\mathcal{S}(k)$ is to be 0 if $\binom{\alpha}{\sigma}$ is empty, or if $C^\iota_{1,k} = \emptyset$.) Similarly, we set

$$\mathcal{S}(0) \overset{\text{def}}{=} \sum_{\iota \in \binom{\alpha}{\sigma}} \sum_{e^{(0)} \in C^\iota_{0,1}} T(\omega^\iota, x^{(1)}(e^{(0)})) \qquad (2.1.6b)$$

Thus we have the desired decomposition:

$$E_{2,1}P(\sigma)\omega = \sum_{k=o}^{N} \mathcal{S}(k) \qquad (2.1.7)$$

To complete the proof of the theorem, it thus suffices to prove the two following equations:

$$\mathcal{S}(0) = (\sigma_{2,1} + 1)P(\sigma + E_{2,1})\omega \qquad (2.1.8)$$

and (for $1 \le k \le N$)

$$\mathcal{S}(k) = (\sigma_{2,k} + 1)P(\sigma + E_{2,k} - E_{1,k})\omega \qquad (2.1.9)$$

First, suppose $1 \leq k \leq N$, and consider $\mathcal{S}(k)$. In proving (2.1.9), we must consider separately two cases, according to whether or not $\sigma + E_{2,k} - E_{1,k}$ has all entries non-negative.

CASE ONE: $\underline{\sigma + E_{2,k} - E_{1,k} \in \Pi^N}$

In order to establish (2.1.9), we next construct a $(\sigma_{2,k}+1)$-to-one correspondence, mapping the

$$[\#\binom{\sigma}{\alpha}] \cdot \sigma_{1,k}$$

terms

$$\{T(\omega^\iota, x^{(k)}(e^{(k)}) : \iota \in \binom{\alpha}{\sigma}, e^{(k)} \in C^\iota_{1,k}\}$$

of the sum (2.1.6a) which makes up $\mathcal{S}(k)$, into the

$$\#\binom{\sigma + E_{2,k} - E_{1,k}}{\alpha}$$

terms

$$\{\omega^\kappa : \kappa \in \binom{\sigma + E_{2,k} - E_{1,k}}{\alpha}\}$$

whose sum is $P(\sigma + E_{2,k} - E_{1,k})\omega$, in such a way that equality holds for each pair of corresponding terms.

Namely, for every pair

$$\iota \in \binom{\alpha}{\sigma}, e^{(k)} \in C^\iota_{1,k} \tag{2.1.10}$$

let us define

$$\kappa = \Phi_k(\iota, e^{(k)}) = \{C^\kappa_{i,j} : i, j \in \underline{N}\}$$

to be the $(\sigma + E_{2,k} - E_{1,k})$-selection specified by

$$\begin{cases} C^\kappa_{2,k} = C^\iota_{2,k} \cup \{e^{(k)}\} \\ C^\kappa_{1,k} = C^\iota_{1,k} - \{e^{(k)}\} \\ C^\kappa_{i,j} = C^\iota_{i,j}\,, & \text{otherwise.} \end{cases} \tag{2.1.11}$$

(In other words, κ is obtained from ι by moving $e^{(k)}$ from $C^\iota_{1,k}$ to $C^\iota_{2,k}$.)

Then, for the selection κ thus defined, (1.4.6) obviously becomes

$$\begin{cases} (\omega^\kappa)_1 = x^{(1)}[[\underline{a_1} - \{e^{(k)}\} - C^\iota_{2,1} - \ldots - C^\iota_{N,1}]] \cdot x^{(2)}[[C^\iota_{1,2}]] \cdot \ldots \cdot x^{(N)}[[C^\iota_{1,N}]] \\ (\omega^\kappa)_2 = x^{(1)}[[\{e^{(k)}\} \cup C^\iota_{2,1}]] \cdot x^{(2)}[[\underline{a_2} - C^\iota_{1,2} - C^\iota_{3,2} - \cdots - C^\iota_{N,2}]] \cdot \prod_{l=3}^{N} x^{(l)}[[C^\iota_{2,l}]] \\ (\omega^\kappa)_l = C^\iota_{i,j} = (\omega^\iota)_l \text{ if } 3 \leq l \leq N \end{cases}$$

which implies (using (2.1.3), (2.1.3a) and (2.1.4)) that

$$T(\omega^{\iota}, x^{(k)}(e^{(k)}) = \omega^{\kappa}$$

Given any

$$\kappa \in \binom{\alpha}{\sigma + E_{2,k} - E_{1,k}}$$

there are precisely $\sigma_{2,k} + 1$ pairs (2.1.10) such that

$$\kappa = \kappa(\iota, e^{(k)})$$

—given by letting $e^{(k)}$ range through the

$$(\sigma + E_{2,k} - E_{1,k})_{2,k} = \sigma_{2,k} + 1$$

elements of $C^{\kappa}_{2,k}$, and then taking, for each such $e^{(k)}$, the C^{ι}'s uniquely determined by (2.1.11). Hence we obtain

$$\begin{aligned}\mathcal{S}(k) &= \sum_{\iota \in \binom{\alpha}{\sigma}} \sum_{e^{(k)} \in C^{\iota}_{1,k}} T(\omega^{\iota}, x^{(k)}(e^{(k)})) \\ &= (\sigma_{2,,k} + 1) \sum_{\kappa \in \binom{\alpha}{\sigma + E_{2,k} - E_{1,k}}} \omega^{\kappa} \\ &= (\sigma_{2,k} + 1) P(\sigma + E_{2,k} - E_{1,k})\omega\end{aligned}$$

(as was to be proved.)

CASE TWO: $\underline{\sigma + E_{2,k} - E_{1,k} \notin \Pi^N}$

Since $\sigma \in \Pi^N$ by hypothesis, it must be that $\sigma_{1,k} = 0$, whence S(k), being an empty sum, is equal to zero.Since also $P(\sigma + E_{2,k} - E_{1,k}) = 0$, (2.1.9) again holds trivially in this case.

Thus (2.1.9) holds in all cases. We have still to prove (2.1.8). The proof is essentialy the same, with these two variations:
We must replace (2.1.10) by

$$\iota \in \binom{\alpha}{\sigma}, e^{(0)} \in C^{\iota}_{1,0} = \underline{a_1} - C^{\iota}_{1,1} - C^{\iota}_{1,2} - \ldots - C^{\iota}_{1,N} \tag{2.1.10a}$$

and we must replace (2.1.11) by

$$\text{(2.1.11a)} \qquad \begin{cases} C^{\kappa}_{2,1} = C^{\iota}_{2,1} \cup \{e^{(0)}\} \\ C^{\kappa}_{i,j} = C^{\iota}_{i,j} \text{ otherwise} \end{cases}$$

noting that for each

$$\kappa \in \binom{\alpha}{\sigma + E_{21}}$$

there are precisely $\sigma_{21} + 1$ pairs (2.1.10a) such that (2.1.11a) holds (given by letting $e^{(0)}$ range through the $\sigma_{21} + 1$ elements of $C^{\kappa}_{2,1}$.)

This completes the proof of the theorem.

Definition 2.1.2. *An N-shift σ will be called* **reduced** *if all its diagonal entries $\sigma_{i,i}$ vanish (for $1 \le i \le N$). The* **reduced form** *σ^{red} of any N-shift σ, is the N-shift specified by*

$$(\sigma^{\mathrm{red}})_{i,j} = \begin{cases} \sigma_{i,j}, & \text{if } i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.3. *Let V be a module over a commutative ring R. Suppose $a_1, \ldots, a_N$ are non-negative integers,and that*

$$\sigma \in \Pi^N, \omega \in \; S^{a_1}V \otimes \cdots \otimes S^{a_N}V$$

Then

$$P(\sigma)\omega = \prod_{i=1}^{N} \binom{a_i - \sum_{j \neq i} \sigma_{j,i}}{\sigma_{i,i}} \cdot P(\sigma^{\mathrm{red}})\omega \tag{2.1.12}$$

[Note: For an example of this proposition, cf. Example 1.3.2]

PROOF: Let us compare

$$P(\sigma)\omega = \sum_{\iota \in \binom{\alpha}{\sigma}} \omega^{\iota}, \;\text{ with }\; P(\sigma^{\mathrm{red}})\omega = \sum_{\kappa \in \binom{\alpha}{\sigma^{red}}} \omega^{\kappa}\,.$$

We have the epimorphism

$$\phi : \binom{\alpha}{\sigma} \to \binom{\alpha}{\sigma^{\mathrm{red}}}$$

which maps a σ-selection ι into the σ^{red}-selection

$$\kappa = \phi(\iota) = \{C^{\kappa}_{i,j}\}$$

defined by

$$C^{\kappa}_{i,j} = \begin{cases} C^{\iota}_{i,j}, & \text{if } i \neq j \\ \emptyset & \text{if } i = j. \end{cases}$$

Clearly $\omega^\kappa = \omega^\iota$.

Given

$$\kappa \in \binom{\alpha}{\sigma^{\mathrm{red}}}$$

we obtain all ι with $\phi(\iota) = \kappa$ by selecting as $C^\iota_{i,i}$, for each i from 1 to N, an arbitrary $\sigma_{i,i}$-element subset of

$$\underline{a_i} - \bigcup_{\substack{1 \le j \le N \\ j \ne i}} C^\kappa_{j,i}\,.)$$

Thus, ϕ is an M-to-1 map, where

$$M = \prod_{i=1}^N \binom{a_i - \sum_{j\ne i} \sigma_{j,i}}{\sigma_{i,i}},$$

and so we have, as asserted,

$$P(\sigma)\omega = \sum_{\iota \in \binom{\alpha}{\sigma}} \omega^\iota = M \cdot \sum_{\kappa \in \binom{\alpha}{\sigma^{\mathrm{red}}}} \omega^\kappa = MP(\sigma^{\mathrm{red}})\omega\,.$$

Theorem 2.1.4. *Let $\sigma \in \Pi^N$ and let $1 \le i \le N$. Then*

$$E_{i,i}P(\sigma) = (\sigma_{i,i}+1)P(E_{i,i}+\sigma) + (\sum_{k=1}^N \sigma_{i,k})P(\sigma) \tag{2.1.13}$$

(It is to be noted that the coefficients in (2.1.1) and (2.1.13) do not depend, as one might have expected, on $a_1, \cdots, a_N$, but only on σ.)

PROOF:

It suffices, by the same symmetry argument as before, to prove the special case

$$E_{1,1}P(\sigma)\omega = (\sigma_{1,1}+1)P(E_{1,1}+\sigma)\omega + (\sum_{k=1}^N \sigma_{1,k})P(\sigma)\omega \tag{2.1.14}$$

(where ω is given by (1.4.1)).

Let us now set

$$P' \stackrel{\mathrm{def}}{=} \prod_{i=2}^N \binom{a_i - \sum_{j\ne i} \sigma_{j,i}}{\sigma_{i,i}} \cdot P(\sigma^{\mathrm{red}})\omega$$

Note that (by (1.4.4))

$$\underline{a}_1 - \sigma_{2,1} - \cdots - \sigma_{N,1} = \sigma_{1,1} + \sigma_{0,1}$$

Hence, Prop.2.1.3 implies

$$P(\sigma)\omega = \binom{a_1 - \sigma_{2,1} - \cdots - \sigma_{N,1}}{\sigma_{1,1}} P' = \binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1}} P' \tag{2.1.15}$$

Replacing σ by $\sigma + E_{1,1}$ in this last equation yields

$$P(\sigma + E_{1,1})\omega = \binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1} + 1} P' \tag{2.1.16}$$

Recall that $E_{1,1}$ acts on

$$S^{a_1}V \otimes \cdots \otimes S^{a_N}V \tag{2.1.17}$$

as multiplication by a_1. Since ω lies in (2.1.17), it follows by (1.3.6) (which as noted applies both for Def. 1.3.1 and for Def.1.3.2) that

$$P(\sigma)\omega \in S^{a_1 + wt_1(\sigma)}V \otimes \cdots \otimes S^{a_N + wt_N(\sigma)}V$$

Here $wt_i(\sigma)$ is defined by (1.3.5a); we note in particular

$$\begin{aligned} a_1 + wt_1(\sigma) &= a_1 - (\sigma_{2,1} + \cdots + \sigma_{N,1}) + (\sigma_{1,2} + \cdots + \sigma_{1,N}) \\ &= \sigma_{0,1} + \sum_{k=1}^{N} \sigma_{1,k} \end{aligned} \tag{2.1.18}$$

From the preceding, we readily deduce

$$E_{1,1}P(\sigma)\omega = [a_1 - (\sigma_{21} + \cdots + \sigma_{N,1}) + (\sigma_{1,2} + \cdots + \sigma_{1,N}] \binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1}} P' \tag{2.1.19}$$

In the identity

$$(B+1)\binom{A}{B+1} = (A-B)\binom{A}{B} \tag{2.1.20}$$

replace A by $\sigma_{11} + \sigma_{01}$, B by σ_{11}. We obtain

$$(\sigma_{1,1} + 1)\binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1} + 1} = \sigma_{0,1}\binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1}}$$

whence (using also eqns.15, 16 and 19:)

$$\begin{aligned} E_{1,1}P(\sigma)\omega &= (\sigma_{0,1} + \sum_{k=1}^{N} \sigma_{1,k}) \binom{\sigma_{0,0} + \sigma_{0,1}}{\sigma_{1,1}} P' \\ &= [(\sigma_{1,1} + 1)\binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1} + 1} + (\sum_{k=1}^{N} \sigma_{1,k})\sigma_{0,1}\binom{\sigma_{1,1} + \sigma_{0,1}}{\sigma_{1,1}}] \cdot P' \\ &= \sigma_{1,1}P(\sigma + E_{1,1})\omega + (\sum_{k=1}^{N} \sigma_{1,k}))P(\sigma)\omega \end{aligned}$$

which completes the proof of the theorem.

§2.2 The Recurrence for Differential Polarizations

We assume, throughout the present sub-section, that a specific $\mathbb{C}$-basis

$$\mathcal{B} = \{x_1, \ldots, x_M\}$$

has been chosen for V, with all Weyl polarizations defined via Def.1.3.1, by means of $\mathcal{B}$. (Only in the following §2.3 will we prove, using the results of the present sub-section, that the differential Weyl polarizations are basis-independent.)

Actually, for the purposes of the present sub-section, computations go a bit more easily with the *non-normalized* differential Weyl polarizations $P_0(\sigma)$:

Theorem 2.2.1. *Let i and j be distinct integers between 1 and N; let $\sigma \in \Pi^N$.Then the four following equations hold:*

$$E_{i,j}P_0(\sigma) = P_0(E_{i,j} + \sigma) + \sum_{k=1}^{N} \sigma_{j,k} P_0(\sigma + E_{i,k} - E_{j,k}) \tag{2.2A}$$

$$P_0(\sigma)E_{i,j} = P_0(E_{i,j} + \sigma) + \sum_{k=1}^{N} \sigma_{k,j} P_0(\sigma + E_{k,j} - E_{k,i}) \tag{2.2B}$$

$$E_{i,i}P_0(\sigma) = P_0(E_{i,i} + \sigma) + \left(\sum_{k=1}^{N} \sigma_{i,k}\right) P_0(\sigma) \tag{2.2C}$$

$$P_0(\sigma)E_{i,i} = P_0(E_{i,i} + \sigma) + \left(\sum_{k=1}^{N} \sigma_{k,j}\right) P_0(\sigma) \tag{2.2D}$$

PROOF:

Suppose

$$\sigma = E_{i_1,j_1} + \cdots + E_{i_L,j_L}\ , \tag{2.2.1}$$

with all i's and j's in $\underline{N}$ (so that $\sigma_{i,j}$ is the number of λ between 1 and L for which $(i_\lambda, j_\lambda) = (i, j)$).

For every

$$F \in \mathbb{C}[X_1^{(1)}, \cdots, X_M^{(1)}; \cdots; X_1^{(N)}, \cdots, X_M^{(N)}] = S^{\otimes N}V\ ,$$

we have (by eqn.(1.3.3))

$$E_{i,j}P_0(\sigma)F = (\sum_{k=1}^{M} X_k^{(i)} \frac{\partial}{\partial X_k^{(j)}}) \sum_{k_1,\cdots,k_L \in \underline{M}} X_{k_1}^{(i_1)} \cdots X_{k_L}^{(i_L)} \frac{\partial^L F}{\partial X_{k_1}^{(j_1)} \cdots \partial X_{k_L}^{(j_L)}}$$

which is in turn equal to the sum $A+B$ of

$$A = \sum_{k,k_1,\cdots,k_L \in \underline{M}} X_k^{(i)} X_{k_1}^{(i_1)} \cdots X_{k_L}^{(i_L)} \frac{\partial^{L+1} F}{\partial X_k^{(j)} \partial X_{k_1}^{(j_1)} \cdots \partial X_{k_L}^{(j_L)}}$$

and

$$B = \sum_{1 \leq \lambda \leq L} \sum_{k_1,\cdots,k_L \in \underline{M}} \delta_{i_\lambda,j} X_{k_\lambda}^{(i)} \cdot X_{k_1}^{(i_1)} \cdots \widehat{X_{k_\lambda}^{(i_\lambda)}} \cdots X_{k_L}^{(i_L)} \frac{\partial^L F}{\partial X_{k_1}^{(j_1)} \cdots \partial X_{k_\lambda}^{(j_\lambda)} \cdots \partial X_{k_L}^{(j_L)}}$$

Clearly $A = P_0(E_{i,j}+\sigma)F$, while

$$B = \sum_{\substack{i_\lambda = j \\ 1 \leq \lambda \leq L}} P_0(E_{i_1,j_1} + \cdots + \underbrace{E_{i,j_\lambda}}_{\text{replacing } E_{i_\lambda,j_\lambda}} + \cdots + E_{i_L,j_L})F$$

(in which the selected term E_{i,j_λ} replaces the term $E_{i_\lambda,j_\lambda} = E_{j,j_\lambda}$ in (2.2.1).) Thus,

$$E_{i,j}P_0(\sigma) = P_0(E_{i,j}+\sigma) + \sum_{\substack{i_\lambda = j \\ 1 \leq \lambda \leq L}} P_0(\sigma + E_{i,j_\lambda} - E_{j,j_\lambda}) \tag{2.2.2}$$

Now,given any k between 1 and N, there are precisely $\sigma_{j,k}$ values of λ for which $i_\lambda = j, j_\lambda = k$, and each contributes the same term $P_0(\sigma + E_{i,k} - E_{j,k})$ to the sum on the right side of eqn.(2.2.1); from which eqn.(2.2A) is immediate.

The proof of (2.2B) is precisely similar. For the same reasons, it suffices to examine here only one of (2.2C) and (2.2D); let us choose the first. Then with notation as before,

$$E_{i,i}P_0(\sigma)F = (\sum_{k=1}^{M} X_k^{(i)} \frac{\partial}{\partial X_k^{(i)}}) \sum_{k_1,\cdots,k_L \in \underline{M}} X_{k_1}^{(i_1)} \cdots X_{k_L}^{(i_L)} \frac{\partial^L F}{\partial X_{k_1}^{(j_1)} \cdots \partial X_{k_L}^{(j_L)}}$$

which is the sum $A' + B'$ of

$$A' = \sum_{k,k_1,\cdots,k_L \in \underline{M}} X_k^{(i)} X_{k_1}^{(i_1)} \cdots X_{k_L}^{(i_L)} \frac{\partial^{L+1} F}{\partial X_k^{(i)} \partial X_{k_1}^{(j_1)} \cdots \partial X_{k_L}^{(j_L)}} = P_0(E_{i,i}+\sigma)F$$

and

$$B' = \sum_{1\le\lambda\le L} \sum_{k_1,\cdots,k_L\in\underline{M}} \delta_{i_\lambda,i} X^{(i)}_{k_\lambda} X^{(i_1)}_{k_1}\cdots\widehat{X^{(i_\lambda)}_{k_\lambda}}\cdots X^{(i_L)}_{k_L} \frac{\partial^L F}{\partial X^{(j_1)}_{k_1}\cdots\partial X^{(j_\lambda)}_{k_\lambda}\cdots\partial X^{(j_L)}_{k_L}}$$

$$= \sum_{\substack{i_\lambda=i\\1\le\lambda\le L}} \sum_{k_1,\cdots,k_L\in\underline{M}} X^{(i_1)}_{k_1}\cdots X^{(i_L)}_{k_L} \frac{\partial^L F}{\partial X^{(j_1)}_{k_1}\cdots\partial X^{(j_\lambda)}_{k_\lambda}\cdots\partial X^{(j_L)}_{k_L}}$$

Thus ,

$$B' = \sum_{\substack{i_\lambda=i\\1\le\lambda\le L}} P_0(\sigma)F = \left(\sum_{k=1}^{N}\sigma_{i,k}\right)P_0(\sigma)F$$

and so finally,

$$E_{i,i}P_0(\sigma) = P_0(E_{i,i}+\sigma) + (\sum_{k=1}^{N}\sigma_{i,k})P_0(\sigma)$$

This completes the proof of Th.2.2.1.

We obtain the analogous formulas for the normalized differential polarizations

$$P(\sigma) = \frac{1}{\sigma!}P_0(\sigma)$$

as a trivial consequence of the preceding; note the first and third of the following formulas, correspond precisely with the results in §2.1 for combinatorial polarizations. (Let us note, however, that the proofs of the corresponding results in §2.1 were rather more intricate than those in the present sub-section.)

Corollary 2.2.2. *Let $\sigma \in \Pi^N$ and let i,j be distinct integers between 1 and N.Then the four following equations hold:*

$$E_{i,j}P(\sigma) = (\sigma_{i,j}+1)P(E_{i,j}+\sigma) + \sum_{k=1}^{N}(\sigma_{i,k}+1)P(\sigma+E_{i,k}-E_{j,k}) \qquad (2.2.2A)$$

$$P(\sigma)E_{i,j} = (\sigma_{i,j}+1)P(E_{i,j}+\sigma) + \sum_{k=1}^{N}(\sigma_{k,j}+1)P(\sigma+E_{k,j}-E_{k,i}) \qquad (2.2.2B)$$

$$E_{i,i}P(\sigma) = (\sigma_{i,i}+1)P(E_{i,i}+\sigma) + (\sum_{k=1}^{N}\sigma_{i,k})P(\sigma) \qquad (2.2.2C)$$

$$P(\sigma)E_{i,i} = (\sigma_{i,i}+1)P(E_{i,i}+\sigma) + (\sum_{k=1}^{N}\sigma_{k,i})P(\sigma) \qquad (2.2.2D)$$

PROOF: Dividing both sides of (1.4A) by $\sigma!$, and observing that

$$(\sigma + E_{i,j})! = \prod_{p,q}(\sigma + E_{i,j})_{p,q} = \sigma!(\sigma_{i,j} + 1\,,)$$

and that

$$(\sigma + E_{i,k} - E_{j,k})! = \sigma! \cdot \frac{\sigma_{i,k} + 1}{\sigma_{j,k}}$$

we immediately obtain (2.2.2A).

The proof of (2.2.2B) is precisely similar. Similarly, dividing both sides of (2.2C), (2.2D) by $\sigma!$, we obtain (2.2.2C),(2.2.2D). This completes the proof of Cor.2.2.2.

Finally,as an immediate consequence of the preceding corollary, we obtain the commutators given by:

Corollary 2.2.3. *Let i and j be distinct integers between 1 and N; let $\sigma \in \Pi^N$.Then $[E_{p,q}, P(\sigma)] = A - B$, where*

$$A = \sum_{k=1}^{N}(\sigma_{p,k} + 1)P(\sigma + E_{p,k} - E_{q,k})\,,$$

and

$$B = \sum_{k=1}^{N}(\sigma_{k,q} + 1)P(\sigma + E_{k,q} - E_{k,p})\,,$$

Also, we have

$$[E_{p,p}, P(\sigma)] = \left(\sum_{k=1}^{N}(\sigma_{p,k} - \sigma_{k,q})\right) P(\sigma) \tag{2.2.3}$$

REMARK: Suppose $i \neq j$. If $\sigma_{j,k} = 0$, then the term $P_0(\sigma + E_{i,k} - E_{j,k})$ occurs with zero coefficient in (2.2.A), but $P(\sigma + E_{i,k} - E_{j,k})$ occurs with positive coefficient $\sigma_{i,k} + 1$ in (2.2.2A) (which is a scalar multiple of (2.2A).) To explain this apparent discrepancy, note that here $(\sigma + E_{i,k} - E_{j,k})_{j,k} = -1$, so by the convention adopted in §1.3,

$$P(\sigma + E_{i,k} - E_{j,k}) = 0\,.$$

§2.3 Equivalence of the Two Kinds of Multi-polarizations

Theorem 2.3.1. *Let $\sigma \in \Pi^N$, and let V be a finite- dimensional complex vector-space, with $\mathbb{C}$-basis*

$$\mathcal{B} = \{x_1, \ldots, x_M\}.$$

Let

$$P^{\mathcal{B}}(\sigma, V) \text{ , resp. } P(\sigma, V)$$

denote the natural transformations

$$S^{\otimes N} \to S^{\otimes N}$$

constructed respectively in Definition 1.3.1 (of differential Weyl polarizations) and Definition 1.3.2 (of combinatorial Weyl polarizations). Then,

$$P^{\mathcal{B}}(\sigma, V) = P(\sigma, V) \tag{2.3.1}$$

PROOF:

By the **weight** $W(\sigma)$ of an N-shift σ, will be meant the sum of all its entries:

$$W(\sigma) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j}$$

We now shall prove (2.3.1) by an induction on $W(\sigma)$:

Assume first: $\underline{W(\sigma) = 1}$

Here there exist i and j in $\underline{N}$ such that $\sigma = E_{i,j}$, and so (2.3.1) becomes the assertion that (as already remarked in §1.2) the two definitions for the elementary polarizations $D_{i,j} = P(E_{i,j})$ coincide whenever they both make sense.

Assume next : $\underline{1 < W(\sigma)}$ **and (2.3.1) holds for all N-shifts of weight $< W(\sigma)$**

There exist i,j in $\underline{N}$ such that $\sigma_{i,j} > 0$; we consider two cases, according as $i \neq j$ or $i = j$.

Suppose first that i and j are distinct.We may write

$$\sigma = \sigma' + E_{i,j}$$

with σ' effective. Replacing σ by σ' in eqns. (2.1.1) and (2.2A) we obtain the two equations

$$\sigma_{i,j} P(\sigma, V) = E_{i,j} P(\sigma', V) - \sum_{k=1}^{N} (\sigma'_{i,k} + 1) P(\sigma' + E_{i,k} - E_{j,k}, V),$$

$$\sigma_{i,j} P^{\mathcal{B}}(\sigma, V) = E_{i,j} P^{\mathcal{B}}(\sigma', V) - \sum_{k=1}^{N} (\sigma'_{i,k} + 1) P^{\mathcal{B}}(\sigma' + E_{i,k} - E_{j,k}, V)$$

All N-shifts on the right sides of these equations, have weights one less than that of σ, so the induction hypothesis implies these right sides are equal. Hence also the left sides are equal. Since $\sigma_{i,j}$ is by assumption non-zero, (2.3.1) follows in the present case.

Suppose next that $i = j$. Replacing σ by the effective N-shift

$$\sigma' = \sigma - E_{i,i}$$

in eqns.(2.1.13) and (2.2C), we obtain the two equations

$$\sigma_{i,i} P(\sigma, V) = E_{i,i} P(\sigma', V) - \sum_{k=1}^{N} (\sigma'_{i,k}) P(\sigma', V)\,,$$
$$\sigma_{i,i} P^{\mathcal{B}}(\sigma, V) = E_{i,j} P^{\mathcal{B}}(\sigma', V) - \sum_{k=1}^{N} (\sigma'_{i,k}) P^{\mathcal{B}}(\sigma', V)$$

and conclude the argument as before.

This completes the proof of Theorem 2.3.1.

From this point on, it is no longer necessary (over the ground-field $\mathbb{C}$) to distinguish between the two types of multi-polarizations.

The following useful corollary is an immediate consequence of the preceding argument:

Corollary 2.3.2. *Let σ be an N-shift of weight $W > 1$, and let i,j in $\underline{N}$ be such that $\sigma_{i,j}$ is non-zero. Then there exist: effective N-shifts (all of weights W-1)*

$$\sigma_1, \sigma_2, \cdots, \sigma_L$$

(for some non-negative integer L, which may be 0) and positive integers m_k $(1 \le k \le L)$ such that

$$\sigma_{i,j} \cdot P(\sigma) = E_{i,j} P(\sigma - E_{i,j}) - \sum_{k=1}^{L} m_k P(\sigma_k)\,.$$

If, in addition, σ has the property

$$\text{for all } i, j \in \underline{N}\,,\ \sigma_{i,j} \neq 0 \Rightarrow i > j\,.$$

then $\sigma_1, \cdots, \sigma_L$ may be chosen, all also to have this property.

This in turn immediately implies:

Corollary 2.3.3. *For every N-shift σ there exists unique $P'(\sigma)$ in $\mathfrak{A}_N$ such that, for every complex vector-space V, the Weyl polarization*

$$P(\sigma, V) : S^{\otimes N}V \to S^{\otimes N}V$$

coincides with multiplication by $P'(\sigma)$.

Chapter 3 APPLICATIONS OF WEYL POLARIZATIONS

§3.1 DIFFERENTIALS IN THE ZELEVINSKY COMPLEX

As promised in the introduction, we shall now utilize the Weyl polarizations, to obtain new and quite explicit formulas, for the differentials in certain natural complexes which (in Zelevinsky's metaphor, [Zel,p.152]) 'materialize' the Jacobi-Trudi identity. This application involves a train of thought spanning roughly a century and a half, and one which has (it would seem) not yet reached its full conclusion, despite relevant work by (*inter alia*) Akin[Akin1,2], Buchsbaum, Doty[Doty and Doty2], Lascoux[Las], Maliakas[Mal], Nielsen[Nielsen], Verma[Verma1,2], Woodcock[Wood] and Zelevinsky[Zel].

Before discussing the *differentials* in the Zelevinsky complex, we must begin by discussing the individual *terms*:

Let us rewrite the Jacobi-Trudi identity (1) in the Introduction, as

$$s_\alpha = \sum_{\pi \in \mathfrak{S}_N} sgn(\pi) \prod_{i=1}^{N} h_{a_i - i + \pi(i)} \tag{3.1.1}$$

Reading this in the Grothendieck ring $GL_{\mathbb{C}}{}^{\wedge}$, s_α corresponds to the Weyl functor $V \mapsto V^\alpha$ (in the terminology of Carter and Lusztig,[CL]) while the symmetric polynomial

$$\prod_{i=1}^{N} h_{a_i - i + \pi(i)} = h_{a_1 - 1 + \pi(1)} h_{a_2 - 2 + \pi(2)} \cdots h_{a_N - N + \pi(N)}$$

occurring in eqn.(3.1.1), is the character of the functor

$$V \mapsto S^{a_1 - 1 + \pi(1)}V \otimes S^{a_2 - 2 + \pi(2)}V \otimes \ \cdots \otimes S^{a_N - N + \pi(N)}V \tag{3.1.2}$$

(where as usual, $SV = \oplus S^i V$ denotes the symmetric algebra on the finite-dimensional complex vector-space V.) It will be convenient to denote by $\mathcal{SYM}^\alpha V$, the vector space

$$S^{a_1}V \otimes S^{a_2}V \otimes \cdots \otimes S^{a_N}V \ ,$$

so that (3.1.2) may thus be denoted by $\mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V$—where we set

$$\rho = (1, 2, \cdots, N) \text{ and } \pi(\rho) = (\pi(1), \pi(2), \cdots, \pi(N))\,.$$

All this suggests that, (temporarily forgetting about the differentials), we may hope to obtain a complex $ZEL(\alpha) = ZEL(\alpha, V)$ in $GL_{\mathbb{C}}{}^{\wedge}$, the alternating character-sum of whose terms corresponds as desired to the sum in (3.1.1), if we define the i-th term of this complex to be

$$ZEL^i(\alpha, V) = \bigoplus_{\substack{\pi\in\mathfrak{S}_N \\ l(\pi)=i}} \mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V \tag{3.1.3}$$

(where $l(\pi)$ denotes the number of inversions of a permutation π in $\mathfrak{S}_N$.) It will be convenient to denote the individual summands in (3.1.3) by

$$ZEL^{\pi}(\alpha, V) \stackrel{\text{def}}{=} \mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V = S^{a_1-1+\pi(1)}V \otimes S^{a_2-2+\pi(2)}V \otimes\ \cdots \otimes S^{a_N-N+\pi(N)}V \tag{3.1.3a}$$

Thus we are led (all the authors cited above appear to agree on this) to seek for differentials d_k which will render exact the following complex:

$$ZEL(\alpha, V) : \cdots \xrightarrow{d_{k+1}} ZEL^k(\alpha, V) \xrightarrow{d_k} \cdots \xrightarrow{d_2} ZEL^1(\alpha, V) \xrightarrow{d_1} ZEL^0(\alpha, V) \tag{3.1.4}$$

and for which d_1 has cokernel the Weyl module V^{α} Here we also wish to require the differentials d_k to be natural transformations—something required by all but one of the authors cited.

There is a further generalization in ([Zel]); namely, Zelevinsky next drops the requirement that the N-tuple

$$\alpha = (a_1, \cdots, a_N)$$

represents a partition, requiring only that the a's be integers— the complex (3.1.4) constructed by Zelevinsky remains exact in this greater generality (while if in addition α is a partition, then $\operatorname{coker}(d_1) = V^{\alpha}$). The remainder of the present discussion is to be understood in this greater generality.

To specify the differentials

$$d_l : ZEL^l(\alpha, V) = \bigoplus_{\substack{\pi\in\mathfrak{S}_N \\ l(\pi)=l}} \mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V \to ZEL^{l-1}(\alpha, V) = \bigoplus_{\substack{\pi'\in\mathfrak{S}_N \\ l(\pi')=l-1}} \mathcal{SYM}^{\alpha-\rho+\pi'(\rho)}V$$

is the same, as to specify the collection of C-linear transformations

$$d_\alpha^{\pi,\pi'} : \mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V = S^{a_1+\pi(1)-1}V \otimes S^{a_2+\pi(2)-2}V \otimes \cdots \otimes S^{a_N+\pi(N)-N}V$$
$$\to \mathcal{SYM}^{\alpha-\rho+\pi'(\rho)}V = S^{a_1+\pi'(1)-1}V \otimes S^{a_2+\pi'(2)-2}V \otimes \cdots \otimes S^{a_N+\pi'(N)-N}V$$

for all

$$\pi, \pi' \in \mathfrak{S}_N \text{ with } l(\pi) = l,\, l(\pi') = l-1 \tag{3.1.5}$$

(We omit l from our notation for these maps, since by (3.1.5) l is determined by π.)

We divide the construction of the maps $d_\alpha^{\pi,\pi'}$ into four smaller parts, as follows:

A. We must specify, for precisely which pairs π, π' the map $d_\alpha^{\pi,\pi'}$ is to be non-zero.

B. To each such pair π, π' meeting condition A, Akin assigns a signature $\pm$, according to the rule to be described below.

C. When π, π' meet condition A, we shall express $d_\alpha^{\pi,\pi'}$ as a $\mathbb{C}$-linear combination of Weyl polarizations $P(\sigma)$—thus we must specify precisely which $P(\sigma)$ appear in this linear combination. This condition will be independent of α, and such shift-matrices σ will be called *subordinate* to the pair π, π'.

D. Finally, when $P(\sigma)$ has been designated as occurring in $d_\alpha^{\pi,\pi'}$, we must specify the precise numerical coefficient with which it occurs. (It will depend on α, and will be an integer if all a_i are integers—indeed, will be a product of certain binomial coefficients and factorials, as specified below.)

A. WHEN IS $d_\alpha^{\pi,\pi'}$ NON-ZERO?

Let (3.1.5) hold.

In the complex to be constructed here (which will be proved later to coincide with that constructed by Zelevinsky) the partial maps $d_\alpha^{\pi,\pi'}$ are defined to be 0 unless π' precedes π in the Bruhat order (This *Ansatz* was suggested to the author by Verma in a conversation.)

It is well known (cf., for example, [Mathas,p.2,Prop.1.3]) that this holds if and only if there exist i,j such that

$$1 \le i < j \le N, \pi(i) > \pi(j),\, \pi' = \pi(i,j) \tag{3.1.6}$$

We shall call (π, π') an **arrow-pair** if these conditions (3.1.5), (3.1.6) are satisfied. (The relation of such arrow-pairs to the theory of maps between Verma modules of A_N will be discussed below in §3.4)

B. AKIN'S NORMALIZATION OF THE BGG SIGNATURE

Let $\mathcal{A}_{\mathcal{N}}$ denote the set of all arrow-pairs of $\mathfrak{S}_N$, as defined above. In [BGG], it is proved that there exists a map

$$s : \mathcal{A}_{\mathcal{N}} \to \{1, -1\}$$

with this property:

For every set of four arrow-pairs

$$w_1 \overset{(i_1,j_1)}{\longrightarrow} w_2, w_2 \overset{(i_2,j_2)}{\longrightarrow} w_4, w_1 \overset{(i_3,j_3)}{\longrightarrow} w_3, w_3 \overset{(i_4,j_4)}{\longrightarrow} w_4, \tag{3.1.7}$$

in $\mathcal{A}$, **we have**

$$s(w_1, w_2)s(w_2, w_4)s(w_1, w_3)s(w_3, w_4) = -1 \tag{3.1.8}$$

We shall call such a map a BGG-**signature** for $\mathcal{A}_{\mathcal{N}}$. (Although the treatment in [BGG], and so some of the following discussion in this section, make sense in much greater generality, for our present purposes it suffices to restrict to the consideration of $\mathcal{A}_{\mathcal{N}}$.) [BGG] calls a configuration such as (3.1.7) a **square**; we denote (3.1.7) by the diagram

$$\begin{array}{ccc} w_1 & \underset{(i_1,j_1)}{\longrightarrow} & w_2 \\ \downarrow{\scriptstyle (i_3,j_3)} & & \downarrow{\scriptstyle (i_2,j_2)} \\ w_3 & \underset{(i_4,j_4)}{\longrightarrow} & w_4 \end{array} \tag{3.1.9}$$

More generally, if $\mathcal{A}'_{\mathcal{N}}$ is a subset of $\mathcal{A}_{\mathcal{N}}$, by a **partial** BGG-**signature for** $\mathcal{A}'_{\mathcal{N}}$, will be meant a map

$$s : \mathcal{A}'_{\mathcal{N}} \to \{1, -1\}$$

such that (3.1.8) holds for every square (3.1.9), for which all four arrow-pairs lie in $\mathcal{A}'_{\mathcal{N}}$.

The BGG resolutions, for the case A_N, are only completely specified once one of the (many) BGG signatures for $\mathcal{A}_{\mathcal{N}}$ has been selected. Because Zelevinsky's construction in [Zel] of the complex studied in the present section is obtained from the BGG resolution, it also is only specified up to the choice of a BGG signature.

This ambiguity is perhaps not very serious, but Akin in [Akin1,2] has shown one specific way to choose these matters in a completely unique fashion —let us pause here to sketch his normalization, since it is fairly short, and the author has not seen it presented as an explicit algorithm in the literature. Akin defines a specific BGG-signature, as follows:

For every permutation $w \in \mathfrak{S}_N$, let $P(w)$ denote the set of permutations $\overline{w}$ which precede w in the Bruhat order—i.e., such that there exists a chain

$$w = w_0 \stackrel{(i_0,j_0)}{\longrightarrow} w_1 \longrightarrow \cdots \longrightarrow w_l \stackrel{(i_l,j_l)}{\longrightarrow} \overline{w}$$

of arrow-pairs in $\mathfrak{S}_N$, beginning with w and ending with $\overline{w}$. (We include w in $P(w)$.)

Let

$$\tilde{w} = \begin{pmatrix} 1, 2, \cdots, N-1, N \\ N, N-1, \cdots, 2, 1 \end{pmatrix} : i \mapsto N+1-i$$

denote the element in $\mathfrak{S}_N$ of maximal length $\binom{N}{2}$. (Note that thus $P(\tilde{w}) = \mathfrak{A}_N$ and $P(I) = \{I\}$.)

Let Σ denote a chain

$$\Sigma : \tilde{w} = w_{\binom{N}{2}} \to \ldots \to w_0 = I \tag{3.1.10}$$

of $\binom{N}{2}$ arrow-pairs in $\mathfrak{S}_N$, such that each arrow-pair

$$w_{p+1} \stackrel{(i_p,i_p+1)}{\longrightarrow} w_p$$

in the chain Σ is associated with an *elementary* transposition (i_p, i_p+1).

Then as Akin notes implicitly, the *existence proof* in [BGG], actually gives a *constructive algorithm* for computing— from the datum Σ—a BGG-signature s_Σ for $\mathfrak{S}_N$, by the inductive procedure next to be explained.

Suppose that $0 \leq p < \binom{N}{2}$, and that a partial BGG-signature s_p has been given for $P(w_p)$. Then the argument in ([BGG],pp.56 and 57) in fact shows that the following rules furnish a well-defined extension of s_p (involving no choices other than that of Σ) to a partial BGG-signature s_{p+1} for $P(w_{p+1})$:

Let $w \stackrel{(q,r)}{\longrightarrow} w'$ be an arrow-pair, with w (hence w') in $P(w_{p+1})$. In order to compute $s_{p+1}(w, w')$, we must consider four cases:

$\underline{\mathrm{Case\,I : w \in P(w_p)}}$

Then also $w' \in P(w_p)$ Since we wish s_{p+1} to extend s_p, we are forced to define

$$s_{p+1}(w, w') \stackrel{\text{def}}{=} s_p(w, w')$$

$\underline{\mathrm{Case\,II : w \notin P(w_p), (q,r) = (i_p, i_p+1)}}$

Here [BGG] defines $s_{p+1}(w,w')$ to be $+1$.(This *Ansatz* implies the rules for the two following Cases.)

$\underline{\mathrm{Case\,III}: \mathrm{w} \notin \mathrm{P}(\mathrm{w_p}), \mathrm{w}' \notin \mathrm{P}(\mathrm{w_p}), (\mathrm{q},\mathrm{r}) \neq (\mathrm{i_p}, \mathrm{i_p}+1)}$

$\underline{\mathrm{Case\,IV}: \mathrm{w} \notin \mathrm{P}(\mathrm{w_p}), \mathrm{w}' \in \mathrm{P}(\mathrm{w_p}), (\mathrm{q},\mathrm{r}) \neq (\mathrm{i_p}, \mathrm{i_p}+1)}$

In both of these two cases, it follows from ([BGG], Lemma 11.3 on p.53), that

$$w \xrightarrow{(i_p,i_p+1)} w(i_p, i_p+1)$$

and

$$w' \xrightarrow{(i_p,i_p+1)} w'(i_p, i_p+1)$$

are arrow-pairs, and that $w(i_p, i_p+1)$ and $w'(i_p, i_p+1)$ lie in $P(w_p)$.

Thus, in Case III, $s_{p+1}(w,w')$ is well-defined by

$$s_{p+1}(w,w') \stackrel{\mathrm{def}}{=} -s_p(w(i_p,i_p+1), w'(i_p,i_p+1))$$

(i.e., by the requirement, that the product of the edge-signatures of the following square

$$\begin{array}{ccc} w & \longrightarrow & w' \\ \big\downarrow{\scriptstyle +1} & & \big\downarrow{\scriptstyle +1} \\ w(i_p,i_p+1) & \longrightarrow & w'(i_p,i_p+1) \end{array}$$

is to be -1.)

Similarly, in Case IV we must set

$$s_{p+1}(w,w') \stackrel{\mathrm{def}}{=} -s_p(w(i_p,i_p+1), w'(i_p,i_p+1)) s_p(w', w'(i_p,i_p+1))$$

Continuing inductively, this algorithm gives the desired BGG-signature $s_{N(N-1)/2} = s_\Sigma$ on $P(\tilde{w}) = \mathcal{A}_N$

We shall (following Akin) utilize a canonical choice of the chain Σ, which is perhaps sufficiently explained by giving the case $N=4$:

$$1234 \longrightarrow \underbrace{1243}_{\text{float }3} \longrightarrow \underbrace{1423 \longrightarrow 1432}_{\text{float }2} \longrightarrow \underbrace{4132 \longrightarrow 4312 \longrightarrow 4321}_{\text{float }1} \tag{3.1.11}$$

The associated BGG-signature will be called the *Akin signature*, and will be denoted by sgn_A.

C. THE SHIFT-MATRICES SUBORDINATE TO AN ARROW-PAIR

Let (π,π') be an arrow-pair, so in particular there exist unique integers i,j satisfying (3.1.6). We define the *multiplicity* of (π,π') to be the positive integer

$$r = \pi(i) - \pi(j) > 0 \tag{3.1.12}$$

We shall define $d_\alpha^{\pi,\pi'}$ to be a suitable $\mathbb{Z}$-linear combination of those Weyl polarizations $P(\sigma)$, for which σ lies in the set defined as follows:

Definition 3.1.1. *Let $1 \leq i < j \leq N$, and let r be a positive integer. Then we denote by $TERM(i,j,r)$ the set of all N-shifts*

$$\sigma \in \Pi^N$$

which satisfy the following three conditions:

I) $\sigma_{p,q} = 0$ *unless* $i \leq q < p \leq j$.

II) $r = \sum_{l=i+1}^{j} \sigma_{l,i}$

III) For all k strictly between i and j, we have

$$\sum_{l=i}^{j} \sigma_{l,k} = \sum_{l=i}^{j} \sigma_{k,l} \tag{3.1.13}$$

(i.e., $\sigma_{k,i} + \cdots + \sigma_{k,k-1} = \sigma_{k+1,k} + \cdots + \sigma_{j,k}$.)

Also, it will be convenient to denote by $R_k(\sigma)$ the common value of both sides of eqn.(3.1.13).

In particular, if i, j, r are related as above to an arrow-pair (π, π') , then we shall say that the elements of the set $TERM(i,j,r)$ just defined, are *subordinate to* (π, π').

Let us again note specifically, that this condition on σ is totally independent of α, and indeed only depends on (i,j) and r.

D. THE NUMERICAL COEFFICIENT OF $P(\sigma)$ IN $d_l^{\pi,\pi'}$

Let (π, π') be an arrow-pair, with i, j determined by (2.2), and with multiplicity $r = \pi(i) - \pi(j)$. Let

$$\alpha = (a_1, ..., a_N) \in \mathbb{C}^N ,$$

and let σ be an N-shift subordinate to (π, π').

Definition 3.1.2. *Under the preceding hypotheses, we define the* **amplitude**

$$\langle \sigma; \pi, \pi' \rangle \in \mathbb{C}$$

to be the integer given by

$$\langle \sigma; \pi, \pi' \rangle = r! \cdot \prod_{k=i+1}^{j-1} \left[R_k(\sigma)!(r - R_k(\sigma))! \binom{\pi(i) - \pi(k)}{r - R_k(\sigma)} \right] \tag{3.1.14}$$

(with the understanding the product in (3.1.14) is to be taken equal to 1 if it is empty, i.e. if $j = i + 1$.)

Note:This amplitude is independent of α, depending (as the notation indicates) only on the N-shift σ and the arrow-pair (π, π'). This amplitude is an integer, which may be negative.

CONSTRUCTION OF THE MAPS $d_\alpha^{\pi,\pi'}$

We may now, finally, complete our construction of the natural transformations $d_\alpha^{\pi,\pi'}$, and hence of the differentials in the complex (3.1.5), as follows:

Let (π, π') be an arrow-pair, with i, j determined by (2.2), and with multiplicity $r = \pi(i) - \pi(j)$. Let

$$\alpha = (a_1, ..., a_N) \in \mathbb{Z}^N ,$$

let V be a complex vectorspace, and let sgn be a BGG-signature for $\mathcal{A}_N$(not necessarily the Akin signature). Finally, let

$$\omega \in ZEL^\pi(V, \alpha) = S^{b_1}V \otimes S^{b_2}V \otimes \cdots \otimes S^{b_N}V$$

(with the b's determined by (3.1.3a))

Then we define

$$d_{\alpha,sgn}^{\pi,\pi'}\omega \stackrel{\text{def}}{=} sgn(\pi, \pi') \cdot \sum_{\sigma \in TERM(i,j,r)} \langle \sigma; \pi, \pi' \rangle \cdot P(\sigma)\omega \tag{3.1.15}$$

This completes our **construction** of the complex $ZEL(\alpha, V)$; we must postpone till Chapter 4 the **proof** of the following theorem, which asserts that the result of this construction is indeed is an exact sequence, (except possibly for its last term) and coincides completely with the complex obtained by the method of Zelevinsky:

Theorem 3.1.3. *With*

$$d_\alpha^{\pi,\pi'} = d_{\alpha,sgn}^{\pi,\pi'}$$

given by (3.1.15), let us define the maps d_k in (3.1.4) by

$$d_k = \bigoplus_{\substack{l(\pi)=k \\ l(\pi')=k-1}} d_\alpha^{\pi,\pi'} ;$$

then these maps coincide completely with the maps

$$d_k^{ZEL} : \bigoplus_{\substack{\pi \in \mathfrak{S}_N \\ l(\pi)=k}} \mathcal{SYM}^{\alpha-\rho+\pi(\rho)}V \to \bigoplus_{\substack{\pi' \in \mathfrak{S}_N \\ l(\pi')=k-1}} \mathcal{SYM}^{\alpha-\rho+\pi'(\rho)}V$$

constructed by Zelevinsky in [Zel] from the BGG-resolutions for A_N (constructed using sgn.)

Let us conclude this section by once again emphasizing, that the preceding construction makes **NO** use of the theory of Verma modules. By contrast, the **proof** this construction yields an exact sequence—at least, the only complete proof available at present to the author—makes heavy use of the work of Verma, Bernstein-Gel'fand-Gel'fand, Shapovalov and Zelevinsky.

We shall next consider some specific examples of the construction just explained.

§3.2 **SOME ILLUSTRATIVE EXAMPLES**

EXAMPLE 3.2.1

Our first example involves the case N=3. The special case $N = 3$ of the Zelevinsky complex (with α a partition) is presented in ([Doty], pp.134–136), where this result is attributed to Verma. In Doty's complex, the differentials are explicitly furnished as elements of $\mathfrak{A}_3$, but not, of course, expressed in the language of multi-polarizations. Thus, this earlier data provides an excellent test for our assertions. (Although it is assumed in [Doty] that α is a partition, in fact these results are valid without the assumption $a_1 \geq a_2 \geq a_3 \geq 0$.) As we shall see, the complex presented for $N = 3$ by Doty and Verma, is in fact precisely the Zelevinsky complex, normalized by the choice of the Akin signature sgn_A.

Assume then,

$$N = 3\,, \alpha = (a_1, a_2, a_3) \in \mathbb{Z}^3$$

and let V denote a finite-dimensional complex vector-space. It will be convenient to denote the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \pi 1 & \pi 2 & \pi 3 \end{pmatrix}$$

by $[\pi 1\, \pi 2\, \pi 3]$.

Here the Zelevinsky complex $ZEL(\alpha, V)$ assumes the form

$$0 \to ZEL^3 \xrightarrow{d_3} ZEL^2 \xrightarrow{d_2} ZEL^1 \xrightarrow{d_1} ZEL^0 \tag{1}$$

with the terms ZEL^l given by (3.1.3) and (3.1.4) as follows:

$$\begin{aligned} ZEL^0 &= S^{a_1}V \otimes S^{a_2}V \otimes S^{a_3} \\ ZEL^1 &= ZEL[213] \oplus ZEL[132] \\ &= (S^{a_1+1}V \otimes S^{a_2-1}V \otimes S^{a_3}V) \oplus (S^{a_1}V \otimes S^{a_2+1}V \otimes S^{a_3-1}V) \\ ZEL^2 &= ZEL[231] \oplus ZEL[312] \\ &= (S^{a_1+1}V \otimes S^{a_2+1}V \otimes S^{a_3-2}V) \oplus (S^{a_1+2}V \otimes S^{a_2-1}V \otimes S^{a_3-1}V) \\ ZEL^3 &= ZEL[321] = S^{a_1+2}V \otimes S^{a_2}V \otimes S^{a_3-2}V \end{aligned}$$

Having thus computed the *terms* in (1), let us next turn to the more interesting question of the *differentials*. To obtain these, let us go in order through the four steps explained in §3.1:

<u>STEP A :</u> $\mathcal{A}_3$ consists of the 8 arrow-pairs:

$$\begin{cases} \tau_1 : [213] \xrightarrow{(1,2)} [123], \tau_2 : [132] \xrightarrow{(2,3)} [123], \tau_3 : [231] \xrightarrow{(1,3)} [132], \tau_4 : [231] \xrightarrow{(2,3)} [213], \\ \tau_5 : [312] \xrightarrow{(1,2)} [132], \tau_6 : [312] \xrightarrow{(1,3)} [213], \tau_7 : [321] \xrightarrow{(1,2)} [231], \tau_8 : [321] \xrightarrow{(2,3)} [312] \,. \end{cases} \tag{3.2.1}$$

<u>STEP B :</u> We must next compute the Akin signature sgn_A for these 8 arrow-pairs. For chain (3.1.10) with $N{=}3$, we take the Akin choice

$$w_3 = [321] \xrightarrow{(2,3)} w_2 = [312] \xrightarrow{(1,2)} w_1 = [132] \xrightarrow{(2,3)} w_0 = I$$

from which the inductive procedure of BGG yields the Akin signatures given in the following table (where the arrow-pairs τ_i are given by (3.2.1)):

i	1	2	3	4	5	6	7	8
$sgn_A(\tau_i)$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$

(For readers interested in applying the inductive algorithm of §3.1.1 to verify this table, it may be helpful to note that, setting

$$F_i = P(w_i) \backslash P(w_{i+1})\,,$$

there is induced on $\mathfrak{S}_3$ the filtration given by:

$$F_0 = \{[123]\}, F_1 = \{[132]\}, F_2 = \{[213], [312]\}, F_3 = \{[231], [321]\}\ )$$

<u>STEPS C and D :</u> Here the maps

$$d_\alpha^{\pi,\pi'} = d_\alpha(\tau)$$

(where τ denotes the arrow-pair $\pi \to \pi'$) which are furnished by the remaining two steps, are given in the second row of the following table, while the third row lists the maps $d_{\mathrm{DV}}(\tau)$ furnished by Doty and Verma; it is asserted that the second and third rows coincide, i.e., that $d_\alpha(\tau) = d_{\mathrm{DV}}(\tau)$.

i	1,8	2,7	3	4	5	6
$d_\alpha(\tau_i)$	$E_{2,1}$	$E_{3,2}$	$E_{3,1} - P(E_{3,2} + E_{2,1})$	$2P(2E_{3,2})$	$2P(2E_{2,1})$	$-2E_{3,1} - P(E_{3,2} + E_{2,1})$
$d_{\mathrm{DV}}(\tau_i)$	$E_{2,1}$	$E_{3,2}$	$E_{3,2}E_{2,1} - 2E_{2,1}E_{3,2}$	$(E_{3,2})^2$	$(E_{2,1})^2$	$E_{2,1}E_{3,2} - 2E_{3,2}E_{2,1}$

We chase through the details for the case

$$\tau_3 : [231] \xrightarrow{(1,3)} [132]\,;$$

it is left as an exercise to any reader so interested, to verify the other entries in this table by the same straightforward algorithm.

The natural transformation

$$\begin{aligned} d_\alpha(\tau_3) : ZEL[2,3,1] = S^{a_1+1}V \otimes S^{a_2+1}V \otimes S^{a_3-2}V \\ \to ZEL[1,3,2] = S^{a_1}V \otimes S^{a_2+1}V \otimes S^{a_3-1}V \end{aligned} \tag{3.2.2}$$

is one of the four constituents of d_2.

Here

$$\pi = \begin{pmatrix}123\\231\end{pmatrix}, \pi' = \begin{pmatrix}123\\132\end{pmatrix}, (i,j) = (1,3)$$

Hence by (3.1.8), $r = \pi(1) - \pi(3) = 1$. Examination of Def.3.1.1 (in Case C of the preceding subsection) shows that the 3-shift σ is subordinate to π and π', (i.e., lies in $TERM(1,3,1)$), precisely when:

(i) All $\sigma_{i,j}$ are 0, except possibly $\sigma_{2,1}, \sigma_{3,1}$, and $\sigma_{3,2}$.

(ii) $\sigma_{2,1} + \sigma_{3,1} = 1$

(iii) $\sigma_{2,1} = \sigma_{3,2}$

Hence

$$TERM((1,3,1) = \{\sigma_1, \sigma_2\}$$

with

$$\sigma_1 = E_{3,1}\,, \sigma_2 = E_{2,1} + E_{3,2}$$

Thus, $d_\alpha(\tau_3)$ is a $\mathbb{C}$-linear combination of $P(2E_{3,1})$ and $P(E_{2,1} + E_{3,2})$, with the numerical coefficients next to be determined (via Def.3.1.2 in Step D of the preceding subsection.)

Consider first

$$\sigma_1 = E_{3,1} \ .$$

Here

$$R_2(\sigma_1) = 0 \ ,$$

and eqn.(3.1.15) yields the amplitude

$$< E_{3,1}; \pi, \pi' >= 1!0!1! \binom{\pi(1) - \pi(2)}{1} = \binom{-1}{1} = -1$$

Similarly, we have

$$R_2(\sigma_2) = 1 \ ,$$

and so

$$< E_{3,2} + E_{2,1}; \pi, \pi' >= 1$$

Recalling that $sgn_A(\tau_3) = -1$, (3.1.16) gives the entry

$$d_\alpha(\tau_3) = E_{3,1} - P(E_{3,2} + E_{2,1}) \tag{3.2.3a}$$

in the preceding table.

It remains to verify that this map coincides, on their common domain

$$ZEL[2,3,1] = S^{a_1}V \otimes S^{a_2+1}V \otimes S^{a_3-1}V \ ,$$

with the Doty-Verma map

$$d_{\mathrm{DV}}(\tau_3) = [E_{3,2}E_{2,1} - 2E_{2,1}E_{3,2}] | ZEL[2,3,1] \tag{3.2.3b}$$

To prove this, it suffices to verify that both maps have the same effect on every element

$$\omega = (x_1 \cdots x_{a_1+1}) \otimes (y_1 \cdots y_{a_2+1}) \otimes (z_1 \cdots z_{a_3-2})$$

in

$$ZEL[231] = S^{a_1+1}V \otimes S^{a_2+1}V \otimes S^{a_3-2}V$$

(all x's, y's and z's in V). Let us set

$$A := E_{3,1}\omega = \sum_{i=1}^{a_1+1} (x_1 \cdots \widehat{x_i} \cdots x_{a_1+1}) \otimes (y_1 \cdots y_{a_2+1}) \otimes (x_i \cdot z_1 \cdots z_{a_3-2})$$

and

$$B := P(E_{3,2} + E_{2,1})\omega =$$
$$= \sum_{i=1}^{a_1+1} \sum_{j=1}^{a_2+1} (x_1 \cdots \widehat{x_i} \cdots x_{a_1+1}) \otimes (x_i \cdot y_1 \cdots \widehat{y_j} \cdots y_{a_2+1}) \otimes (y_j \cdot z_1 \cdots z_{a_3-2})$$

(Note that A and B both lie in $ZEL[1,3,2] = S^{a_1}V \otimes S^{a_2+1}V \otimes S^{a_3-1}V$.)

Then direct computation (as explained in §1.2) shows that

$$E_{3,2}E_{2,1}\omega = E_{3,2} \sum_{i=1}^{a_1} (x_1 \cdots \widehat{x_i} \cdots x_{a_1+1}) \otimes (x_i \cdot y_1 \cdots y_{a_2+1}) \otimes (z_1 \cdots z_{a_3-2}) = A + B$$

and similarly

$$E_{2,1}E_{3,2}\omega = B$$

Hence, finally,

$$d_{\mathrm{DV}}(\tau_3)\omega = (A+B) - 2B = A - B = E_{3,1}\omega - P(E_{3,2} + E_{2,1})\omega = d_\alpha(\tau_3)\omega \; .$$

(Question: Is there some simple rule one could use directly to compute the Akin signature—or some other specific BGG-signature—rather than the tedious step-by-step inductive algorithm presented in §3.1?)

EXAMPLE 3.2.1

Our next example involves $N = 4$ and the arrow-pair

$$\tau : [2341] \xrightarrow{(2,4)} [2143]$$

Let sgn denote an arbitrary choice (not necessarily sgn_A) of BGG-signature on $\mathcal{A}_4$.

If $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$, and V varies over complex vector-spaces, then the natural transformation $d_{\alpha,V}(\tau)$, which maps

$$ZEL^{[2341]}(\alpha, V) = S^{a_1+1}V \otimes S^{a_2+1}V \otimes S^{a_3+1}V \otimes S^{a_4-3}V$$

into

$$ZEL^{[2143]}(\alpha, V) = S^{a_1+1}V \otimes S^{a_2-1}V \otimes S^{a_3+1}V \otimes S^{a_4-1}V \; ,$$

is a component of the differential d_3 in $ZEL(\alpha, V)$ (since [2341] has 3 inversions.) This mapping is, in fact, that resulting from the action on the $\mathfrak{A}_4$-module $ZEL^{[2341]}(\alpha, V)$ of

the following element in $\mathfrak{A}_4$ (which, it should be noted, is completely independent both of α and of V):

$$d(\tau) = sgn(\tau) \cdot [4P(2E_{3,2} + 2E_{4,3}) - 2P(E_{3,2} + E_{4,3} + E_{4,2}) + 2P(2E_{4,2})] \qquad (3.2.5)$$

(The reader may find it a helpful exercise, to compute (3.2.5), using the algorithm explained in §3.1.)

One final bit of propaganda for the efficacy and appropriateness of the Weyl polarizations in such computations: let us examine the precise effect of (3.2.5) on the generating element

$$\omega = \underline{w} \otimes \underline{x} \otimes \underline{y} \otimes \underline{z} = (w_1 \cdots w_{a_1+1}) \otimes (x_1 \cdots x_{a_2+1}) \otimes (y_1 \cdots y_{a_3+1}) \otimes (z_1 \cdots z_{a_4-3})$$

for $ZEL^{[2341]}(\alpha, V)$:

Using eqn.(3.2.4) we obtain

$$d_{\alpha,V}(\tau)\omega = sgn(\tau)(4A - 2B + 2C)$$

where we have set:

$$A = P(2E_{3,2} + 2E_{4,3})\omega = \sum_{\substack{i<i' \\ i,i' \in \underline{a_2+1}}} \sum_{\substack{j<j' \\ j,j' \in \underline{a_3+1}}} \underline{w} \otimes \frac{\underline{x}}{x_i \cdot x_{i'}} \otimes (x_i \cdot x_{i'} \cdot \frac{\underline{y}}{y_j \cdot y_{j'}}) \otimes (y_j \cdot y_{j'} \cdot \underline{z}) \, ,$$

$$B = P(E_{3,2} + E_{4,3} + E_{4,2})\omega = \sum_{\substack{i \neq i' \\ i,i' \in \underline{a_2+1}}} \sum_{j \in \underline{a_3+1}} \underline{w} \otimes \frac{\underline{x}}{x_i \cdot x_{i'}} \otimes (x_i \cdot \frac{\underline{y}}{y_j}) \otimes (x_{i'} \cdot y_j \cdot \underline{z})$$

and

$$C = P(2E_{4,2})\omega = \sum_{\substack{i<i' \\ i,i' \in \underline{a_2+1}}} \underline{w} \otimes \frac{\underline{x}}{x_i \cdot x_{i'}} \otimes \underline{y} \otimes (x_i \cdot x_{i'} \cdot \underline{z})$$

Note that A, B, C all lie, as they ought to, in

$$ZEL^{[2143]}(\alpha, V) = S^{a_1+1}V \otimes S^{a_2-1}V \otimes S^{a_3+1}V \otimes S^{a_4-1}V \, ,$$

§3.3 The action on $S^{\otimes N}$ of the Verma-Shapovalov Elements for A_N

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, with $\mathfrak{h}$ a selected Cartan subalgebra; also assume selected an ordering for $(\mathfrak{g}, \mathfrak{h})$, with Δ^+ the set of positive roots. Let ρ denote half

the sum of the roots in Δ^+. For any positive root α, we denote the associated co-root in $\mathfrak{h}$ by h_α —so that

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$$

for all $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{N}_+$ denote the nilpotent sub-algebra of $\mathfrak{g}$ generated by the positive root spaces, and $\mathfrak{N}_-$ that generated by the negative root spaces.

For any $\lambda \in \mathfrak{h}^*$, we denote by $\mathcal{I}_\lambda$, the left ideal in the enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of $\mathfrak{g}$, generated by

$$\Delta^+ \cup \{h - \lambda(h) \cdot 1 : h \in \mathfrak{h}\}$$

Thus the $\mathfrak{g}$-module

$$\mathcal{V}_\lambda := \mathfrak{A}(\mathfrak{g})/\mathcal{I}_\lambda$$

is precisely the Verma module over $\mathfrak{g}$, with highest weight λ. We denote by v_λ the image of 1 in this quotient, (so that $\mathcal{V}_\lambda$ is cyclic on the distinguished highest-weight vector v_λ.)

Definition 3.3.1. *By a* **Verma triple** *for such a $\mathfrak{g}$, will be meant an ordered triple (α, r, λ) satisfying the following four conditions:*

(i) α *is a positive root in* $(\mathfrak{g}, \mathfrak{h})$.

(ii) $\lambda \in \mathfrak{h}^*$

(iii) r *is a positive integer*

(iv) $\lambda - s_\alpha \bullet \lambda = r\alpha$

REMARK: Here the symbol $\bullet$ designates the "twisted action" of the Weyl group on $\mathfrak{h}$, given by

$$s_\alpha \bullet \lambda = s_\alpha(\lambda + \rho) - \rho$$

Thus condition (iv) may be replaced by the equivalent condition

$$(\lambda + \rho)h_\alpha = r$$

It is a well-known result, due to Verma, that given such a triple, there exists a non-zero $\mathfrak{g}$-linear homomorphism

$$\mathcal{V}_{\lambda - r\alpha} \to \mathcal{V}_\lambda \tag{3.3.1}$$

unique up to scalar multiples. However, since our purpose here is to express this map as an explicit $\mathbb{Z}$-linear combination of (the actions of) Weyl polarizations, it is necessary to select an explicit **normalization** of this element; Shapovalov has constructed one method for doing so, as follows:

Definition 3.3.2. *Let $\tau{=}(\alpha, r, \lambda)$ be a Verma triple for $\mathfrak{g}$.*
By the **Verma-Shapovalov element**

$$\gamma = VS(\tau) = VS_{\alpha,r}(\lambda)$$

for this triple, is meant the element γ in $\mathfrak{A}(\mathfrak{N}_-)$ uniquely specified by the two following conditions:

VS1)There is a non-zero $\mathfrak{g}$-linear map

$$\phi : \mathcal{V}_{\lambda - r\alpha} \to \mathcal{V}_\lambda$$

such that

$$\phi(v_{\lambda - r\alpha}) = \gamma v_\lambda$$

VS2)Choose a total ordering $<<$ of Δ_+; say

$$\{\alpha_1 << \alpha_2 << \cdots << \alpha_m\}, \text{ where } m = \#(\Delta^+)$$

For each $\alpha \in \Delta_+$ let E_α be an associated root vector.

This choice associates to every map $\pi : \Delta^+ \to \mathbb{N}$ a basis element

$$F_\pi := (E_{\alpha_1})^{\pi(\alpha_1)} \cdots (E_{\alpha_m})^{\pi(\alpha_m)} \tag{3.3.2}$$

in the PBW basis for $\mathfrak{A}(\mathfrak{N}_-)$ associated with $<<$, namely the basis

$$\{F_\Pi | \Pi \in (\mathbb{N})^{\Delta^+}\}\,. \tag{3.3.3}$$

Note in particular the distinguished map

$$\pi < r >: \Delta^+ \to \mathbb{N}$$

which maps each of the simple roots to r, and maps all other positive roots to 0. It is then required that, when the Verma-Shapovalov element γ is expanded as a $\mathbb{C}$-linear combination of the PBW-basis (3.3.3), the basis vector $F_{\pi<r>}$ shall have coefficient precisely 1.

REMARKS: The first requirement VS1), specifies $\sigma = VS(\tau)$ uniquely, up to a scalar multiple. (This is the fundamental result of Verma which is the basis of the present paper.) Concerning the second requirement, (due it seems to Shapovalov), whose effect is

to remove this last ambiguity in the definition of VS(τ), let us note that this requirement presupposes two non-obvious facts: 1)The coefficient of $F_{\pi[\tau]}$ is not identically 0 in all operators satisfying VS1). 2)The requirement VS2) is in fact independent of the particular choice of total ordering $<<$ on Δ^+.

These two facts may be found proved, in Franklin([Fra,§3 and 4] * for arbitrary semi-simple Lie algebras, in arbitrary characteristic.

In a recent clarifying discussion of these matters, the present author was informed by Verma, that the construction in Verma's thesis [Verma 1] of a $\mathfrak{g}$-homomorphism (3.3.1) was in fact defined absolutely, not simply up to scalar multiples. This would imply that the construction in Verma's thesis supplied an intrinsic method of normalizing (3.3.1). Below, there will be given a formula (3.3.6), an expression in which will then be verified to satisfy the two Shapovalov conditions listed above, and hence to coincide with the Verma-Shapovalov element.The present author does not know the precise relation between these two methods of normalization (Verma's and Shapovalov's), and hence must here leave open the question of the relations (if any) of formula (3.3.6) below, to the normalization of (3.3.1) propounded by Verma.

From now on, for the remainder of the present paper, we shall restrict ourselves entirely to the Lie algebra

$$\mathfrak{g} = \mathfrak{s}l_N(\mathbb{C}) = A_{N-1}$$

and to its enveloping algebra $(\mathfrak{A}_N)^0$.

In this special case, we choose, (as is usual), the Cartan subalgebra $\mathfrak{h}$, to be that formed by the diagonal $N \times N$ matrices of trace 0. The dual $\mathfrak{h}^*$ of this is spanned over $\mathbb{C}$ by the N linear functionals λ_i (i between 1 and N) where λ_i maps any $N \times N$ matrix C in $\mathfrak{g}$ into its i-th diagonal entry $C_{i,i}$. (Thus the λ_i have sum 0.) The usual choice here for the set of positive roots is

$$\Delta^+ = \{\lambda_i - \lambda_j : 1 \leq i < j \leq N\}$$

and the associated nilpotent subalgebra $\mathfrak{N}_+$ is that formed by the $N \times N$ strictly upper-triangular matrices over $\mathbb{C}$—so that $\mathfrak{N}_-$ is the Lie algebra formed by the strictly lower-triangular ones.

* Caution: There is a typo in the statement of VS2) on p.66 of [Fra]; in (3),loc.cit., $r = \sum n_i\epsilon_i$ must be replaced by $dr = \sum n_i\epsilon_i$

With these choices, it is readily seen that the Verma triples for A_{N-1} consist of those ordered triples

$$(\alpha = \lambda_i - \lambda_j, r, \lambda = \sum_{i=1}^{N} l_i \lambda_i)$$

(with $1 \le i < j \le N, r \in \mathbb{Z}^+$, the l_i being complex numbers), which satisfy

$$l_i - l_j - i + j = r \tag{3.3.4}$$

Definition 3.3.3. *Let*

$$\tau = (\lambda_i - \lambda_j, r, \lambda = \sum_{i=1}^{N} l_i \lambda_i)$$

be a Verma triple for A_{N-1}, and (using Def.3.1.1) let

$$\sigma \in TERM(i, j, r)\,;$$

Then we define the **amplitude** $\langle \sigma; \tau \rangle \in \mathbb{C}$ *to be*

$$\langle \sigma; \tau \rangle = r! \cdot \prod_{k=i+1}^{j-1} \left[R_k(\sigma)!(r - R_k(\sigma))! \binom{l_i - l_k}{r - R_k(\sigma)} \right] \tag{3.3.5}$$

Our goal in the next § will be the proof of:

Theorem 3.3.4. *If*

$$\tau = (\lambda_i - \lambda_j = \alpha, r, \lambda)$$

is a Verma triple for A_{N-1}, and if we set

$$T(\tau) \stackrel{\text{def}}{=} TERM(i, j, r)$$

then

$$VS(\tau) = \sum_{\sigma \in \mathcal{T}(\tau)} \langle \sigma; \tau \rangle P(\sigma)\,. \tag{3.3.6}$$

NOTE: As noted by K.Akin in [Akin2, p.418], the maps

$$d^{\pi,\pi'}_{\alpha,sgn}$$

for the Zelevinsky complex, are given by precisely the same elements

$$sgn(\pi, \pi')VS(\pi, \pi')$$

in $\mathfrak{A}_N$ which furnish the corresponding maps in the BGG-resolution. Hence Th.3.3.4 implies (and is apparently rather stronger than) Th.3.1.3.

Section 4 Proof of the Assertions in Section 3

As observed at the end of §3.3, to prove all the assertions in §3, it suffices to prove Th.3.3.4.

For the rest of this §, there will be assumed the hypotheses of Th. 3.3.4 ; that is, we suppose:

(i) $\tau = (\lambda_i - \lambda_j, r, \lambda = \sum l_i \lambda_i)$ is a Verma triple for $\mathfrak{A}_{N-1}$; and we set

(ii) $\mathcal{T}(\tau) = TERM(i, j, r)$, as defined in Def.3.1.1.

Note that (i) means that:

$1 \leq i < j \leq N$, r is a positive integer, and

$$l_i - l_j - i + j = r \tag{4.1}$$

Then, to complete the proof of Theorem 3.3.4, it suffices to prove that the element γ in $\mathfrak{A}_N$, defined by

$$\gamma \stackrel{\text{def}}{=} \sum_{\sigma \in \mathcal{T}(\tau)} \langle \sigma; \tau \rangle P(\sigma) \tag{4.2}$$

satisfies (relative to the given Verma triple τ) the two conditions VS1) and VS2) which, in Def.3.2.2, characterize the Verma-Shapovalov element $VS(\tau)$.(Here the coefficients

$$\langle \sigma; \tau \rangle$$

are the numbers given by Def.3.3.3.)

Unwrapping all this, we see that the goal of proving Th.3.3.4 will have been achieved, once the three following assertions have been established:

A1 $\gamma \cdot v_\lambda$ is a maximal vector in $\mathcal{V}_\lambda$.

A2 This vector $\gamma \cdot v_\lambda$ has weight $\lambda - r\alpha$.

A3 γ satisfies the normalization condition $VS2$) in Def.3.3.2.

(Note that, if A1 and A2 are satisfied, then the element $\gamma \cdot v_\lambda$ in $\mathcal{V}_\lambda$ generates a sub-module isomorphic to $\mathcal{V}_{\lambda - r\alpha}$, from which VS1) is immediate.)

The next three sub-sections are devoted to the proof, in this order, of these three assertions.

§4.1 Proof That $\gamma \cdot v_\lambda$ is a Maximal Vector in $\mathcal{V}_\lambda$

The purpose of this sub-section is the proof of assertion A1, that is, the proof that:

$$E_{p,p+1} \cdot \gamma \cdot v_\lambda = 0 \text{ for } 1 \leq p \leq N-1 \tag{4.1.1}$$

In the remainder of this paper, $\equiv$ is always to be understood to mean $\equiv$ modulo the left ideal $\mathcal{I}_\lambda$ in $\mathfrak{A}_N$ defined in §3.1. In other words, for all γ and γ' in $\mathfrak{A}_N$,

$$\gamma \equiv \gamma' \text{ if and only if } \gamma \cdot v_\lambda = \gamma' \cdot v_\lambda \,.$$

For instance, eq.(4.1.1) is equivalent to the assertion

$$E_{p,p+1} \cdot \gamma \equiv 0 \tag{4.1.2}$$

This in turn is equivalent to the assertion that

$$[E_{p,p+1}, \gamma] \equiv 0 \text{ for } 1 \leq p \leq N-1 \tag{4.1.3}\,,$$

since by definition every left multiple in $\mathfrak{A}_N$ of $E_{p,p+1}$ lies in $\mathcal{I}_\lambda$.

- Throughout the rest of this sub-section, $1 \leq p < N$.

Our proof of (4.1.3) will proceed in three steps:
We first analyze the commutators

$$[E_{p,p+1}, P(\sigma)]$$

for all σ in $TERM(\tau)$. The results thus obtained will be applied in the second step to expand $[E_{p,p+1}, \gamma]$ as a linear combination of Weyl polarizations with explicitly described integer coefficients. In the third step, we shall finally prove (4.1.3) by showing these coefficients are all 0.

Step One: Analysis of $[E_{p,p+1}, P(\sigma)]$ for $\sigma \in TERM(\tau)$

By Cor. 2.2.3, we have

$$[E_{p,p+1}, P(\sigma)] = A - B \tag{4.1.4}$$

where

$$A = \sum_{k=1}^{N} (\sigma_{p,k} + 1) P(\sigma + E_{p,k} - E_{p+1,k}) \tag{4.1.4a}$$

and

$$B=\sum_{k=1}^{N}(\sigma_{k,p+1}+1)P(\sigma+E_{k,p+1}-E_{k,p}) \tag{4.1.4b}$$

By hypothesis, σ satisfies the conditions of Def. 3.1.1. (in Part C of §3.1). In particular, condition I) of this definition implies that

$$\sigma_{p+1,k}=0 \text{ if } i\leq k<p+1\leq j \text{ does not hold,}$$

in which case

$$(\sigma+E_{p,k}-E_{p+1,k})_{p+1,k}=-1\,,$$

so by eq.(1.4.8),

$$P(\sigma+E_{p,k}-E_{p+1,k})=0\,.$$

Since also $\sigma_{p,p}=0$, we obtain

$$A=1\cdot P(\sigma-E_{p+1,p}+E_{p,p})+\sum_{k=i}^{p-1}(\sigma_{p,k}+1)(P(\sigma-E_{p+1,k}+E_{p,k})$$

In the same way, we have

$$B=P(\sigma-E_{p+1,p}+E_{p+1,p+1})+\sum_{k=p+2}^{j}(\sigma_{k,p+1}+1)P(\sigma-E_{k,p}+E_{k,p+1})$$

Combining the two preceding equations with (4.1.4), we obtain:

$$\begin{aligned}[E_{p,p+1},P(\sigma)]=&[P(\sigma-E_{p+1,p}+E_{p,p})-P(\sigma-E_{p+1,p}+E_{p+1,p+1})]+\\ &+\sum_{k=i}^{p-1}(\sigma_{p,t}+1)(P(\sigma-E_{p+1,k}+E_{p,k})-\\ &-\sum_{k=p+2}^{j}(\sigma_{k,p+1}+1)P(\sigma-E_{k,p}+E_{k,p+1})\end{aligned} \tag{4.1.5}$$

It is next claimed that (4.1.5) is 0 unless $i\leq p<j$:

Indeed, if $p\leq i-1$ then both $\sigma-E_{p+1,p}+E_{p,p}$ and $\sigma-E_{p+1,p}+E_{p+1,p+1}$ are non-effective (since $\sigma_{p+1,p}=0$); moreover, the first sum $\sum_{k=i}^{p-1}$ in (4.1.5) is empty (hence 0), while in the

second sum, each term is 0 (since each $\sigma - E_{k,p} + E_{k,p+1}$ is non-effective)—hence (4.1.5) is 0 in this case. Similarly, (4.1.5) is 0 if $p \geq j$.

We may thus (without loss of generality) strengthen as follows our earlier assumption:

- Throughout the rest of this sub-section, $i \leq p < j$.

For all r between 1 and N, let us set

$$col_r(\sigma) = (r^{th} \text{ column-sum of } \sigma) = \sigma_{1,r} + \sigma_{2,r} + \cdots + \sigma_{N,r}$$

Because of our hypothesis that

$$\sigma \in TERM(\tau)\,,$$

we have

$$col_r(\sigma) = \begin{cases} d, & \text{if } r = i; \\ R_r(\sigma), & \text{if } i < r < j; \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.6}$$

Consider next the first expression

$$F = [P(\sigma - E_{p+1,p} + E_{p,p}) - P(\sigma - E_{p+1,p} + E_{p+1,p+1})] \tag{4.1.7}$$

occurring on the right side of eq.(4.1.5). Since all diagonal entries of σ vanish (because of condition I in Def. 3.2.1), it then follows from Prop.2.2.2 that

$$P(\sigma - E_{p+1,p} + E_{p,p}) = P(\sigma - E_{p+1,p})E_{p,p} - (col_p(\sigma) - 1)P(\sigma - E_{p+1,p})$$

(if $\sigma - E_{p+1,p}$ is effective—but this equation is also valid if $\sigma - E_{p+1,p}$ is not effective, since then both sides are 0)

Similarly, Prop.2.2.2 implies that

$$P(\sigma - E_{p+1,p} + E_{p+1,p+1}) = P(\sigma - E_{p+1,p})E_{p+1,p+1} - (col_{p+1}(\sigma))P(\sigma - E_{p+1,p})$$

Combining these two equations, with the fact that the left ideal $\mathcal{I}_\lambda$ contains (for all k between 1 and N)the elements

$$E_{k,k} - E_{k+1,k+1} - \lambda(E_{k,k} - E_{k+1,k+1}) = E_{k,k} - E_{k+1,k+1} - l_k + l_{k+1}$$

we obtain the following congruence for (4.1.7): modulo $\mathcal{I}_\lambda$,

$$F \equiv (l_p - l_{p+1} - col_p(\sigma) + col_{p+1}(\sigma) + 1)P(\sigma - E_{p+1,p})$$

Inserting this last congruence into (4.1.5),we obtain

$$
\begin{aligned}
[E_{p,p+1}, P(\sigma)] \equiv\ & (l_p - l_{p+1} - col_p(\sigma) + col_{p+1}(\sigma) + 1)P(\sigma - E_{p+1,p}) \\
& + \sum_{k=i}^{p-1} (\sigma_{p,k} + 1)(P(\sigma - E_{p+1,k} + E_{p,k}) - \\
& - \sum_{k=p+2}^{j} (\sigma_{k,p+1} + 1)P(\sigma - E_{k,p} + E_{k,p+1})
\end{aligned} \tag{4.1.8}
$$

Step Two: Expansion of $[E_{p,p+1}, \gamma]$ into Multi-polarizations

We combine (4.1.8), with

$$[E_{p,p+1}, \gamma] = \sum_{\sigma \in TERM(\tau)} \langle \sigma; \tau \rangle [E_{p,p+1}, P(\sigma)]$$

to obtain the congruence

$$[E_{p,p+1}, \gamma] \equiv \mathcal{S}'_p + \mathcal{S}''_p + \mathcal{S}'''_p\,, \tag{4.1.9}$$

where

$$\mathcal{S}'_p = \sum_{\sigma \in TERM(\tau)} \langle \sigma; \tau \rangle (l_p - l_{p+1} - col_p(\sigma) + col_{p+1}(\sigma) + 1)P(\sigma - E_{p+1,p}) \tag{4.1.9a}$$

$$\mathcal{S}''_p = \sum_{\sigma \in TERM(\tau)} \sum_{k=i}^{p-1} \langle \sigma; \tau \rangle (\sigma_{p,k} + 1)P(\sigma - E_{p+1,k} + E_{p,k}) \tag{4.1.9b}$$

and

$$\mathcal{S}'''_p = - \sum_{\sigma \in TERM(\tau)} \sum_{k=p+2}^{j} \langle \sigma; \tau \rangle (\sigma_{k,p+1} + 1)P(\sigma - E_{k,p} + E_{k,p+1}) \tag{4.1.9c}$$

We must next collect coefficients of the various P's in (4.1.9):

Lemma 4.1.1.

$$[E_{p,p+1}, \gamma] \equiv \sum_{\sigma \in TERM(\tau)} \{A_p(\sigma) + B_p(\sigma) + C_p(\sigma)\} P(\sigma - E_{p+1,p}) \tag{4.1.10}$$

where

$$A_p(\sigma) = \sum_{\sigma \in TERM(\tau)} ((l_p - l_{p+1} - col_p(\sigma) + col_{p+1}(\sigma) + 1) \langle \sigma; \tau \rangle \tag{4.1.10a}$$

$$B_p(\sigma) = \sum_{k=i}^{p-1} \sigma_{p,k} \langle \sigma + E_{p+1,k} - E_{p,k} - E_{p+1,p}; \tau \rangle \tag{4.1.10b}$$

and

$$C_p(\sigma) = - \sum_{k=p+2}^{j} \sigma_{k,p+1} \langle \sigma + E_{k,p} - E_{k,p+1} - E_{p+1,p}; \tau \rangle \tag{4.1.10c}$$

Note: As explained earlier, in the present paper we adopt everywhere the two conventions, that

$$\sigma \notin TERM(\tau) \Rightarrow \langle \sigma; \tau \rangle = 0\,.$$

and that

$$\sigma \in \Pi_{\pm}^{N} \text{ and } \sigma \text{ non-effective, imply } P(\sigma) = 0$$

In particular, the preceding equations are to be interpreted in this way. Thus, in the ensuing argument, some caution will be needed, to distinguish the 'regular' terms in these sums, from those which are assigned the value 0 by the conventions just reviewed.

Proof of Lemma 4.1.1: To prove the Lemma, it clearly suffices to verify the following two assertions:

<u>CLAIM ONE</u> : *If $\sigma \in TERM(\tau)$ then $P(\sigma - E_{p+1,p})$ either is 0, or has the same coefficient on both sides of (4.1.10).*

<u>CLAIM TWO</u> : *If $P(\sigma')$ occurs with nonzero coefficient as a term in at least one of the three sums (4.1.10a, b, c), then $\sigma' + E_{p+1,p} \in TERM(\tau)$.*

<u>PROOF OF CLAIM ONE</u> :

Let $\sigma \in TERM(\tau)$. We distinguish two cases:

Case One: $\sigma_{p+1,p} = 0$

Here $\sigma - E_{p+1,p}$ is ineffective, so

$$P(\sigma - E_{p+1,p}) = 0\,,$$

(and so the value chosen for its coefficient cannot affect the validity of (4.1.10)).

Case Two: $\sigma_{p+1,p} \geq 1$

Here $\sigma - E_{p+1,p}$ is effective. Now let us list those terms in the three sums (4.1.9a,b,c) for which the assigned argument inside the symbol P, is precisely $\sigma - E_{p+1,p}$.

<u>Inside (4.1.9a):</u> There occurs one such term, with coefficient given by (4.1.10a).

Inside (4.1.9b): Precisely one such term (i.e., containing $P(\sigma - E_{p+1,p})$) is assigned to each ordered pair (σ', k) with

$$\sigma' \in TERM(\tau), \quad i \leq k < p$$

and such that

$$\sigma' - E_{p+1,k} - E_{p,k} = \sigma - E_{p+1,p}$$

—i.e., such that

$$\sigma' = \sigma(k)' \stackrel{\text{def}}{=} \sigma - E_{p+1,p} + E_{p+1,k} - E_{p,k} \,. \tag{4.1.11b}$$

To be explicit, to each such (σ', k) corresponds the term

$$T_b(\sigma', k) \stackrel{\text{def}}{=} \langle \sigma'; \tau \rangle (\sigma'_{p,k} + 1) P(\sigma - E_{p+1,p}) \,.$$

Thus, if we set

$$F = \{k : i \leq k < p \text{ and } \sigma(k)' \in TERM(\tau)$$

then the sum of the coefficients of $P(\sigma - E_{p+1,p})$ for all such $T_b(\sigma', k)$, is

$$\sum_F \{(\sigma(k)')_{p,k} + 1\} \langle \sigma'(k); \tau \rangle = \sum_F \sigma_{p,k} \langle \sigma(k)'; \tau \rangle \,.$$

But there is no change in the value of this sum if we replace

$$\sum_F \text{ by } \sum_{k=i}^{p-1}$$

since by the conventions explained above, if k is such that

$$\sigma(k)' \notin TERM(\tau) \,,$$

then

$$\langle \sigma(k)'; \tau \rangle = 0 \,.$$

Thus the sum of the coefficients of the terms in question is precisely the sum

$$B_p(\sigma) = \sum_{k=i}^{p-1} \sigma_{p,k} \langle \sigma(k)'; \tau \rangle$$

given by eq.(4.1.10b).

Inside (4.1.9c): Essentially the same argument, shows that the terms in (4.1.9c) which involve $P(\sigma - E_{p+1,p})$, are precisely those of the form

$$T_c(\sigma'', k) \stackrel{\text{def}}{=} -\langle\sigma''; \tau\rangle(\sigma''_{k,p+1} + 1)P(\sigma - E_{p+1,p}),$$

with

$$\sigma'' = \sigma(k)'' \stackrel{\text{def}}{=} \sigma - E_{p+1,p} + E_{k,p} - E_{k,p+1}, \tag{4.1.11c}$$

where $p + 2 \leq k \leq j$ and

$$\sigma(k)'' \in TERM(\tau).$$

As before, these terms have sum $C_p(\sigma)$ given by (4.1.10c).

PROOF OF CLAIM TWO :

This is clear for the terms in (4.1.9a).

Consider next the terms in (4.1.9b). Let

$$T = \langle\sigma'; \tau\rangle(\sigma'_{p,k} + 1)P(\sigma' - E_{p+1,k} + E_{p,k})$$

be a non-zero term in (4.1.9b)—so, in particular, we have:

$$\sigma' \in TERM(\tau), \text{ and } \sigma' - E_{p+1,k} + E_{p,k} \text{ is effective.}$$

Set

$$\sigma = \sigma' - E_{p+1,k} + E_{p,k} + E_{p+1,p}$$

Clearly σ is effective, with $\sigma_{p+1,p} > 0$. Since the weight-vector $wt(\sigma)$ defined by eq.(1.10) is additive, and since

$$\epsilon(E_{p+1,k} + E_{p,k} + E_{p+1,p}) = 0,$$

it follows that

$$\epsilon(\sigma) = \epsilon(\sigma') = r\alpha.$$

Hence σ satisfies conditions II) and III) in Def.3.2.1. It is also immediate from $k < p$ that σ satisfies condition I) in this definition.Hence $\sigma \in TERM(\tau)$,and so

$$\sigma' = \sigma(k)', \text{ and the term T coincides with } T_b(\sigma', k),$$

as was to be proved.

The case of non-zero terms inside (4.1.9c) is precisely similar.

This completes the proof of Lemma 4.1.1.

The following simple lemma embodies an argument used in the preceding, and will find several further applications below.

Lemma 4.1.2. *Assume*

$$i \le p \le j,\, \sigma \in TERM(\tau),\, \sigma_{p+1,p} > 0\,.$$

a)Suppose also $i \le k < p$ *; then we have:*

$$\sigma(k)' \in TERM(\tau) \iff \sigma(k)' \text{ is effective,}$$

where

$$\sigma(k)' \stackrel{\text{def}}{=} \sigma - E_{p+1,p} + E_{p+1,k} - E_{p,k}\,.$$

b)Suppose instead $p+2 \le k < j$*; then we have*

$$\sigma(k)'' \in TERM(\tau) \iff \sigma(k)'' \text{ is effective,}$$

where

$$\sigma(k)'' \stackrel{\text{def}}{=} \sigma - E_{p+1,p} + E_{k,p} - E_{k,p+1}\,.$$

Proof: If $i \le k < p$ then $-E_{p+1,p}+E_{p+1,k}-E_{p,k}$ has excess vector 0; since the excess vector ϵ is additive,

$$\epsilon(\sigma(k)') = \epsilon(\sigma) + 0 = d\alpha\,.$$

The assertion a) is now clear from the Remark in §3.2. The proof of b) is similar.

Step Three End-game: Proof $[E_{p,p+1}, \gamma] = 0$

We are still assuming that $i \le p \le j-1$, and that

$$\tau = (\lambda_i - \lambda_j, r, \sum_{k=1}^{N} l_k \lambda_k)$$

is a Verma triple—whence

$$l_i - l_j - i + j = r\,. \tag{4.1.13}$$

Let G_p denote the set of all σ in $TERM(\tau)$ such that

$$\sigma - E_{p+1,p} \text{ is effective, i.e. such that } \sigma_{p+1,p} \ge 1 \tag{4.1.14}$$

By Lemma 4.1.1, the desired result

$$[E_{p,p+1}, \gamma] = 0$$

will follow, if we show for every σ in G_p, that the expression (4.1.10) equals 0.

Thus, to complete the proof of (4.1.3), it suffices to verify, for all $\sigma \in G_p$, that

$$A_p(\sigma) + B_p(\sigma) + C_p(\sigma) = 0 \tag{4.1.15}$$

where

$$A_p(\sigma) = \{l_p - l_{p+1} - col_p(\sigma) + col_{p+1}(\sigma) + 1\} \cdot \langle \sigma; \tau \rangle$$

$$B_p(\sigma) = \sum_{k=i}^{p-1} \sigma_{p,k} \langle \sigma + E_{p+1,k} - E_{p,k} - E_{p+1,p}; \tau \rangle$$

and

$$C_p(\sigma) = - \sum_{k=p+2}^{j} \sigma_{k,p+1} \langle \sigma + E_{k,p} - E_{k,p+1} - E_{p+1,p}; \tau \rangle$$

From here on, we shall write R_k for

$$R_k(\sigma) = \sum_{l=i}^{j} \sigma_{k,l} = \sum_{l=i}^{j} \sigma_{l,k} = col_k(\sigma)$$

(as given by Def.3.1.1 and by eqn. (4.1.6); and will also set

$$S_k = r - R_k \, .$$

The proof of (4.1.15) divides at this point into three cases, according as

$$p = i, \; i < p < j-1 \text{ or } p = j-1 \, .$$

CASE I: $\underline{p = i}$

Let $\sigma \in G_i$. Here (4.1.14) becomes $\sigma_{i+1,i} > 0$. Since $\sigma \in TERM(\tau)$,

$$col_i(\sigma) = r \text{ and } col_{i+1}(\sigma) = R_{i+1} \, ,$$

so that

$$\begin{aligned} A_i(\sigma) &= (l_i - l_{i+1} - r + R_{i+1} + 1) \cdot \langle \sigma; \tau \rangle \\ &= (l_i - l_{i+1} - S_{i+1} + 1) \cdot r! \cdot \prod_{q=i+1}^{j-1} R_q! S_q! \binom{l_i - l_{i+1} - i + q}{S_q} \end{aligned} \tag{4.1.16}$$

$B_i(\sigma) = 0$ in the present case, since the sum in (4.1.15b) here becomes a sum over the empty indexing set $\{k : i \leq k \leq i-1\}$.

We must next evaluate

$$C_i(\sigma) = -\sum_{k=i+2}^{j} \sigma_{k,i+1} \langle \sigma(k)''; \tau \rangle$$

where, for

$$i + 2 \leq k \leq j \tag{4.1.17}$$

we set, as before,

$$\sigma(k)'' = \sigma + E_{k,i} - E_{k,i+1} - E_{i+1,i} \ .$$

Now,

$$(\sigma(k)'')_{s,t} = \begin{cases} \sigma_{k,i} + 1 & \text{if } (s,t) = (k,i) \\ \sigma_{k,i+1} - 1 & \text{if } (s,t) = (k,i+1) \\ \sigma_{i+1,i} - 1 \geq 0 & \text{if } (s,t) = (i+1,i) \\ \sigma_{s,t} & \text{otherwise} \end{cases}$$

and since σ is effective, it follows that $\sigma(k)''$ is effective, if and only if $\sigma_{k,i+1} > 0$. It is now helpful (still assuming (4.1.17)), to divide into two sub-cases:

<u>Sub-case</u> <u>Ia</u>: $\underline{\sigma_{k,i+1} > 0}$

Here, $\sigma(k)''$ is effective, hence, by Lemma 4.1.2, is in $TERM(\tau)$. Thus, Def.3.2.2 is applicable, and if we set

$$R_q'' = R_q(\sigma(k)''), S_q'' = r - R_q'' \text{ for } i < q < j \, ,$$

we have

$$\langle \sigma(k)''; \tau \rangle = r! \cdot \prod_{q=i+1}^{j-1} (R_q'')!(S_q'')! \binom{l_i - l_q - i + q}{S_q''}$$

We next compute the integers R_q'', S_q'' in this formula:

Since $\sigma(k)'' \in TERM(\tau)$,

$$R_q'' = \sum_{s=1}^{N} (\sigma(k)'')_{q,s} = \sum_{s=i}^{q-1} (\sigma(k)'')_{q,s}$$

Thus,

$$R_{i+1}'' = (\sigma(k)'')_{i+1,i} = \sigma_{i+1,i} + 0 - 0 - 1 = R_i - 1, \; S_{i+1}'' = d - R_{i+1}'' = S_i + 1 \, ;$$

while for $i+2 \leq q \leq j-1$,

$$R''_q = (\sigma(k)'')_{q+1,q} + \cdots + (\sigma(k)'')_{j,q} = \sigma_{q+1,q} + \cdots + \sigma_{j,q} = R_q,\ S''_q = S_q\,.$$

Thus

$$\langle \sigma(k)''; \tau \rangle = (R_{i+1}-1)!(S_{i+1}+1)!\binom{l_i - l_{i+1}+1}{S_{i+1}+1} \cdot T \tag{4.1.18}$$

where

$$T = r! \cdot \prod_{q=i+2}^{j-1} R_q! S_q! \binom{l_i - l_q - i + q}{S_q}\,. \tag{4.1.18a}$$

This of course trivially implies

$$\sigma_{k,i+1} \cdot \langle \sigma(k)''; \tau \rangle = \sigma_{k,i+1} \cdot (R_{i+1}-1)!(S_{i+1}+1)!\binom{l_i - l_{i+1}+1}{S_{i+1}+1} \cdot T \tag{4.1.19}$$

(This last deduction may seem less silly in a second.)

<u>Sub-case</u> <u>Ib:</u> $\underline{\sigma_{k,i+1} = 0}$ Here (4.1.18) is in general false, but (4.1.19) is still valid! Namely, here $\sigma(k)''$ is **not** effective, so by the conventions we are using,

$$\langle \sigma(k)''; \tau \rangle = 0\,.$$

Thus, in general Def.3.2.2, i.e., (4.1.18), need not hold in this case— but both sides of (4.1.19) are here 0.
(Note also that $\langle \sigma(k)''; \tau \rangle$ is independent of k in sub-case Ia), while it takes the (in general different) value 0 in sub-case Ib).)

Thus, we have proved that (4.1.19) holds for all k such that (4.1.17) holds. Together with

$$R_{i+1} = \sum_{k=i+2}^{j} \sigma_{k,i+1}$$

we obtain

$$\begin{aligned} C_i(\sigma) &= -(\sum_{k=i+2}^{j} \sigma_{k,i+1}) \cdot (R_{i+1}-1)!(S_{i+1}+1)!\binom{l_i - l_{i+1}+1}{S_{i+1}+1} \cdot T \\ &= -R_{i+1}!(S_{i+1}+1)!\binom{l_i - l_{i+1}+1}{S_{i+1}+1} \cdot T \end{aligned}$$

Let us rewrite (4.1.16) as

$$A_i(\sigma) = (l_i - l_{i+1} - S_{i+1} + 1) \cdot R_{i+1}! S_{i+1}! \binom{l_i - l_{i+1}+1}{S_{i+1}} \cdot T$$

(with T still given by (4.1.18a)) Thus we have

$$A_i(\sigma) + B_i(\sigma) + C_i(\sigma) =$$
$$R_{i+1}!S_{i+1}!\left[(l_i - l_{i+1} - S_{i+1} + 1)\binom{l_i - l_{i+1} + 1}{S_{i+1}} - (S_{i+1} + 1)!\binom{l_i - l_{i+1} + 1}{S_{i+1}}\right] T$$

which is seen to vanish upon replacing, in the following elementary combinatorial identity, M by $l_i - l_{i+1} + 1$ and N by S_{i+1}:

$$(M - N)\binom{M}{N} = (N + 1)\binom{M}{N+1} \qquad (4.1.20)\,,$$

(valid for M any complex number, N any non-negative integer.)

This proves (4.1.15) when $p = i$.

CASE II: $\underline{i < p < j - 1}$

Let $\sigma \in G_p$. Thus,

$$\sigma \in TERM(\tau), \text{ and } \sigma_{p+1,p} \geq 1\,.$$

Both p and $p + 1$ are strictly between i and j. We have

$$col_p(\sigma) = R_p, \text{ and } col_{p+1}(\sigma) = R_{p+1}\,,$$

and so

$$A_p(\sigma) = (l_p - l_{p+1} - R_p + R_{p+1} + 1) \cdot r! \cdot \prod_{i<q<j} R_q!S_q!\binom{l_i - l_q - i + q}{S_q} \qquad (4.1.21)$$

Consider next

$$B_p(\sigma) = \sum_{k=i}^{p-1} \sigma_{p,k}\langle\sigma(k)'; \tau\rangle$$

where (as before)

$$\sigma(k)' = \sigma + E_{p+1,k} - E_{p,k} - E_{p+1,p}\,.$$

Its evaluation (which, with a few modifications, is quite similar to the evaluation just completed of $C_p(\sigma)$ in Case I) proceeds as follows:

Since $\sigma_{p+1,p} > 0$, and $k < p$, and since σ is effective, we see that

$$\sigma(k)' \text{ is effective} \iff \sigma_{p,k} > 0\,.$$

Accordingly, we now divide the study of $\sigma(k)'$ into two sub-cases, depending on whether or not $\sigma_{p,k}$ is 0.

<u>Sub-case</u> <u>IIa:</u> $\underline{\sigma_{p,k} > 0 \,, i \leq k < p}$

In sub-case IIa), $\sigma(k)'$ is effective, hence, by Lemma 4.2.2, is in $TERM(\tau)$. Thus, if we set

$$R'_q = R_q(\sigma(k)'),\ S'_q = S_q(\sigma(k)') = r - R'_q$$

then (in the present sub-case)

$$\langle \sigma(k)'; \tau \rangle = r! \prod_{q=i+1}^{j-1} (R'_q)!(S'_q)! \binom{l_i - l_q - i + q}{S'_q}$$

In this formula, the integers R'_q, S'_q associated with $\sigma(k)'$ coincide with the same integers for σ, except possibly for $q = p, p+1$ or k; while

$$R'_p = R_p - 1, R_{p+1} = R_{p+1}, R'_t = R_t; \text{ so } S'_p = S_p + 1, S'_{p+1} = S_{p+1}, S'_t = S_t \,.$$

Thus, the amplitude is here given by

$$\langle \sigma(k)'; \tau \rangle = (R_p - 1)!(S_p + 1)! \binom{l_i - l_p - i + p}{S_p + 1} \cdot T' \tag{4.1.22}$$

where

$$T' = r! \cdot \prod_{\substack{q \neq p \\ i < q < j}} R_q! S_q! \binom{l_i - l_q - i + q}{S_q} \tag{4.1.22a}$$

and so of course

$$\sigma_{p,k} \langle \sigma(k)'; \tau \rangle = \sigma_{p,k} (R_p - 1)!(S_p + 1)! \binom{l_i - l_p - i + p}{S_p + 1} \cdot T' \tag{4.1.23}$$

<u>Sub-case</u> <u>IIb:</u> $\underline{\sigma_{p,k} = 0 \,, i \leq k < p}$

(4.1.23) remins valid in the present sub-case, since both sides are 0.

Thus, in Sub-cases IIa) and IIb) alike, (4.1.23) is valid, and since

$$R_p = \sum_{k=i}^{p-1} \sigma_{p,k}$$

it follows that

$$B_p(\sigma) = \sum_{k=i}^{p-1} \sigma_{p,k}(R_p - 1)!(S_p + 1)!\binom{l_i - l_p - i + p}{S_p + 1}T' \tag{4.1.24}$$
$$= R_p!(S_p + 1)!\binom{l_i - l_p - i + p}{S_p + 1}T'$$

We next turn to the computation of

$$C_p(\sigma) = -\sum_{k=p+2}^{j} \sigma_{k,p+1}\langle\sigma(k)'';\tau\rangle$$

where now

$$\sigma(k)'' = \sigma + E_{k,p} - E_{k,p+1} - E_{p+1,p}\,.$$

This computation is similar to the preceding one of $B_p(\sigma)$, with the following modifications:

Assume $p + 2 \le k \le j,\ \sigma \in G_p$.Then

$$\sigma(k)'' \text{ is effective } \iff \sigma_{k,p+1} > 0\,,$$

and we have two sub-cases to consider, according as $\sigma_{k,p+1} > 0$ or not.

If $\sigma_{k,p+1} > 0$ then (by Lemma 4.1.2) $\sigma(k)''$ is in $TERM(\tau)$, so we may use Def.3.2.2 to compute the amplitude $\langle\sigma(k)'';\tau\rangle$. Since $i < s < j$, one readily verifies that

$$R_q(\sigma(k)'') = \begin{cases} R_{p+1} - 1 & \text{if } q = p+1 \\ R_q & \text{if } q \ne p+1 \end{cases}\ ;\ S_q(\sigma(k)'') = \begin{cases} S_{p+1} + 1 & \text{if } q = p+1 \\ S_q & \text{if } q \ne p+1 \end{cases}$$

whence we obtain

$$\langle\sigma(k)'';\tau\rangle = (R_{p+1} - 1)!(S_{p+1} + 1)!\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1} + 1}\cdot T'' \tag{4.1.25}$$

where

$$T'' = r!\prod_{\substack{q \ne p+1 \\ i<q<j}} R_q!S_q!\binom{l_i - l_q - i + q}{S_q} \tag{4.1.25a}$$

Thus, whether $\sigma_{k,p+1} > 0$ or not,

$$\sigma_{k,p+1}\cdot\langle\sigma(k)'';\tau\rangle = \sigma_{k,p+1}\cdot(R_{p+1} - 1)!(S_{p+1} + 1)!\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1} + 1}\cdot T'' \tag{4.1.26}$$

is always valid. Together with

$$R_{p+1} = \sum_{k=p+2}^{j} \sigma_{k,p+1}\,,$$

this yields (as in the preceding discussion of $B_p(\sigma)$)

$$C_p(\sigma) = -R_{p+1}!(S_{p+1}+1)!\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1}+1} \cdot T'' \tag{4.1.27}$$

It is now convenient to introduce the following common divisor of the right-hand sides of eqns.(4.1.21),(4.1.24) and (4.1.27):

$$T = r! \cdot R_p!S_p!(R_{p+1})!(S_{p+1})! \cdot \prod_{\substack{q\neq p, q\neq p+1\\ i<q<j}} R_q!S_q!\binom{l_i - l_q - i + q}{S_q} \tag{4.1.28}$$

Then we have

$$A_p(\sigma) + B_p(\sigma) + C_p(\sigma) = TU \tag{4.1.29}$$

with

$$\begin{aligned} U = {} & (l_p - l_{p+1} - R_p + R_{p+1} + 1)\binom{l_i - l_p - i + p}{S_p} \\ & + (S_p + 1)\binom{l_i - l_p - i + p}{S_p + 1}\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1}} \\ & - (S_{p+1} + 1)\binom{l_i - l_p - i + p}{S_p}\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1} + 1} \end{aligned} \tag{4.1.29a}$$

Thus, to prove (4.1.15), it suffices in the present case to verify that $U = 0$.

For this purpose, first observe

$$l_p - l_{p+1} - R_p + R_{p+1} + 1 = -(l_i - l_p - i + p - S_p) + (l_i - l_{p+1} - i + p + 1 - S_{p+1})$$

so that we may write

$$U = \binom{l_i - l_p - i + p}{S_p}U_1 + \binom{l_i - l_{p+1} - i + p + 1}{S_{p+1}}U_2$$

with

$$U_1 = (l_i - l_{p+1} - i + p + 1 - S_{p+1})\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1}} - (S_{p+1}+1)\binom{l_i - l_{p+1} - i + p + 1}{S_{p+1} + 1}$$

and

$$U_2 = -(l_i - l_p - i + p - S_p)\binom{l_i - l_P - i + p}{S_p} + (S_p + 1)\binom{l_i - l_p - i + p}{S_p + 1}$$

and then observe that U_1 and U_2 both vanish because of (4.1.20).

This proves (4.1.15) when $i < p < j - 1$.

CASE III: $\underline{p = j - 1}$

Remark: Up to this point, the argument has made no use of the Verma condition. In other words, except for the single value $j-1$ for p, (4.1.3) holds with no restrictions on the value of λ. Thus, it can be predicted that we shall need to utilize the Verma condition (4.1), in the present (final remaining) case, i.e. to utilize

$$l_i - l_j - i + j = r\,.$$

Let us begin by noting that

$$col_p(\sigma) = col_{j-1}(\sigma) = R_{j-1},\; col_{p+1}(\sigma) = col_j(\sigma) = 0\,,$$

so

$$A_{j-1}(\sigma) = (l_{j-1} - l_j - R_{j-1} + 1) \cdot r! \cdot \prod_{i<q<j} R_q! S_q! \binom{l_i - l_q - i + q}{S_q} \tag{4.1.30}$$

Next, we note that in the present case,

$$C_p(\sigma) = 0\,,$$

since the sum $\sum_{k=p+2}^{j}$ on the right side of (4.1.15c), is here extended over the empty set $\{k : j+1 \leq k \leq j\}$.

Finally, we must evaluate

$$B_p(\sigma) = B_{j-1}(\sigma) = \sum_{k=i}^{j-2} \sigma_{j-1,k} \langle \sigma(k)'; \tau \rangle$$

with $\sigma(k)'$ defined (for the present value of p, and for $i \leq k \leq j-2$) by

$$\sigma(k)' = \sigma - E_{j,j-1} + E_{j,k} - E_{j-1,k} \text{ (for } i \leq k \leq j-2)$$

CLAIM: For $i \le k \le j-2$,

$$\sigma_{j-1,k}\langle\sigma(k)';\tau\rangle = \sigma_{j-1,k}\cdot(S_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}+1}\cdot T \tag{4.1.31}$$

where

$$T = r!R_{j-1}!S_{j-1}!\prod_{q=i+1}^{j-2}R_q!S_q!\binom{l_i-l_q-i+q}{S_q} \tag{4.1.31a}$$

The proof is much the same as that used before, to derive the value obtained for $C_i(\sigma)$ in Case I, and used in Case II to derive $B_p(\sigma)$ (eqn. 4.1.24), and $C_p(\sigma)$ (eqn.4.1.27). Namely:

If $\sigma_{p-1,k}=0$, then both sides of (4.1.31) are 0. Otherwise, $\sigma(k)'$ is effective, hence lies in $TERM(\tau)$ (by a last use of Lemma 4.2.2), so we may use Def.3.2.2 to compute the amplitude $\langle\sigma(k)';\tau\rangle$. Noting for this purpose, that if $i<s<j$, then we have

$$R_s(\sigma(k)') = \begin{cases} R_{j-1}-1 & \text{if } s=j-1 \\ R_s & \text{if } s<j_1 \end{cases} \quad ;\text{so } S_s(\sigma(k)') = \begin{cases} Sj-1+1 & \text{if } s=j-1 \\ S_s & \text{if } s<j-1 \end{cases}$$

it follows immediately that

$$\langle\sigma(k)';\tau\rangle = (S_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}+1}\cdot T$$

This completes the proof of Claim (4.1.31).

Combining (4.1.31) with

$$R_{j-1} = \sum_{k=i}^{j-2}\sigma_{j-1,k}$$

now yields the desired evaluation

$$B_{j-1}(\sigma) = (S_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}+1}\cdot T$$

Let us also rewrite (4.1.30) as

$$A_{j-1}(\sigma) = (l_{j-1}-l_j-R_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}}\cdot T$$

Then we get

$$A_{j-1}(\sigma)+B_{j-1}(\sigma)+C_{j-1}(\sigma) =$$
$$\left[(l_{j-1}-l_j-R_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}} + (S_{j-1}+1)\binom{l_i-l_{j-1}-i+j-1}{S_{j-1}+1)}\right]\cdot T$$

Thus, in order to prove (4.1.15), in the present case, it is sufficient to verify that the following expression U vanishes:

$$U = (l_{j-1} - l_j - R_{j-1} + 1)\binom{l_i - l_{j-1} - i + j - 1}{S_{j-1}} + (S_{j-1} + 1)\binom{l_i - l_{j-1} - i + j - 1}{S_{j-1} + 1)}$$

Using (4.1.20), we obtain

$$\begin{aligned} U &= \binom{l_i - l_{j-1} - i + j - 1}{S_{j-1}}[(l_{j-1} - l_j - R_{j-1} + 1) + (l_i - l_{j-1} - i + j - 1 - S_{j-1})] \\ &= \binom{l_i - l_{j-1} - i + j - 1}{S_{j-1}}[l_i - l_j - i + j - (R_{j-1} + S_{j-1})] \\ &= \binom{l_i - l_{j-1} - i + j - 1}{S_{j-1}}[l_i - l_j - i + j - r] \end{aligned}$$

which indeed vanishes whenever the Verma condition (4.1)—i.e.

$$l_i - l_j - i + j = r$$

—is satisfied.

We have thus proved eqn.(4.1.15), hence also eqn.(4.1.3), in all three cases.

This completes the proof of the assertion A1.

§4.2 Proof that $\gamma \cdot v_\lambda$ has Weight $\lambda - r\alpha$

We next turn to the proof of A2, which asserts that the element $\gamma \cdot v_\lambda$ in the $\mathfrak{sl}_N$-module $\mathcal{V}_\lambda$ has weight $\lambda - r\alpha$.

Now, for every N-shift σ, it follows from eqn.(1.11) in §1.3, that $P(\sigma)v_\lambda$ has weight

$$wt(\sigma) + wt(v_\lambda) = wt(\sigma) + \lambda\,,$$

where the weight $wt(\sigma)$ is given by eqn.(1.10). Since γ is, by definition, a $\mathbb{Z}$-linear combination of

$$\{P(\sigma) : \sigma \in TERM(i, j, r)\}$$

the desired assertion A2 will be an immediate consequence of the following lemma:

Lemma 4.2.1. *Let $\sigma \in TERM(i,j,r)$;then*

$$wt(\sigma) = -r \cdot (\lambda_i - \lambda_j)$$

PROOF: By hypothesis, σ satisfies the three conditions I),II),III) of Def.3.2.1. It must then be proved that, for all k between 1 and N inclusive,

$$wt_k(\sigma) = -r \cdot (\lambda_i - \lambda_j)(E_{k,k})$$

i.e., that

$$\sum_{l=1}^{N} \sigma_{k,l} - \sum_{l=1}^{N} \sigma_{l,k} = r \cdot (\delta_{j,k} - \delta_{i,k}) \tag{4.2.1}$$

There are four cases to check:

CASE 1: $\underline{k = i}$

Here the first sum in (4.2.1) vanishes, by condition I) of Def.3.2.1, so we are left with

$$\sum_{l=1}^{N} \sigma_{i,l} = r$$

which is precisely condition II).

CASE 2: $\underline{i < k < j}$

Here the equation to be proved is

$$\sum_{l=1}^{N} \sigma_{k,l} - \sum_{l=1}^{N} \sigma_{l,k} = 0$$

which holds (for all k in the given range) by condition III).

CASE 3: $\underline{k = j}$

Here the second sum in (4.2.1) vanishes by Condition I), and we are left to prove:

$$\sum_{l=i}^{j-1} \sigma_{j,l} = r \tag{4.2.2}$$

For $i \leq s < j$ let us set

$$C(s) = \sigma_{s+1,s} + \sigma_{s+2,s} + \cdots + \sigma_{j,s} + \sum_{i \leq p < s < q \leq j} \sigma_{q,p}$$

Note that

$$C(i) = \sum_{i<k\leq j} \sigma_{k,i} = r$$

by condition I), while for $i < s < j$,

$$C(s) - C(s-1) = \sum_{i\leq k<s} \sigma_{s,k} - \sum_{s<k\leq j} \sigma_{s,k}$$

which equals 0 by Condition III) of Def.3.2.1.Hence $C(j-1) = r$, which is the same as (4.2.2).

CASE 4:$\underline{k < i \text{ or } k > j}$

Here both of the sums occurring in (4.2.1) are empty.

This completes the proof of (4.2.1), and hence of Assertion A2.

§4.3 Proof that γ satisfies the Shapovalov normalization condition VS2)

Let us define an N-shift σ to be *lower-triangular* if it has this property:

$$\text{for all } k, l \in \underline{N}\,,\ \sigma_{k,l} \neq 0 \Rightarrow k > l\,.$$

Denote by $(\Pi^N)^{LT}$ the set of all such. (Note that all the $P(\sigma)$ involved in the formula (4.1) for γ have lower-triangular σ.)

Choose (arbitrarily) a total ordering $<<$ for Δ^+, say

$$\alpha_1 << \cdots << \alpha_m$$

where

$$\alpha_s = (i(s), j(s)) \text{ with } i(s) < j(s)\,(1 \leq s \leq m)$$

This in turn induces a total ordering (which by a slight abuse of notation will also be denoted by $<<$) on the set of all lower-triangular N-shifts, defined as follows:

To each lower-triangular N-shift σ assign the ordered m-tuple

$$\langle\sigma\rangle \overset{\text{def}}{=} (\sigma_{j(1),i(1)}, \sigma_{j(2),i(2)}, \cdots, \sigma_{j(m),i(m)})$$

Given a second lower-triangular N-shift τ, we shall say that σ precedes τ in the given total ordering,

$$\sigma << \tau\,,$$

if and only if $\langle\sigma\rangle$ precedes $\langle\tau\rangle$ in the usual lexicographic ordering on ordered m-tuples of non-negative integers.

The *weight* $W(\sigma)$ of any N-shift σ is defined to be

$$W(\sigma) \stackrel{\text{def}}{=} \sum_{k,l \in \underline{N}} \sigma_{k,l} \, .$$

As in §3.1, we assign to every lower-triangular N-shift σ, the basis-vector

$$F_\sigma := (E_{j(1),i(1)})^{\sigma(j(1),i(1))} \cdot \cdots \cdot (E_{j(m),i(m)})^{\sigma(j(m),i(m))}$$

in the PBW-basis

$$\mathfrak{B}(<<) = \{F_\sigma \big| \sigma \in (\Pi^N)^{LT}\} \tag{4.3.1}$$

for $\mathfrak{A}(N_-)$ determined by $<<$.

We take the usual filtration for the enveloping algebra $\mathfrak{A}(N_-)$, whereby the s-th filtration-level $\mathfrak{A}(N_-)_s$ is the $\mathbb{C}$-span of all products of $\leq s$ elements $E_{i,j}$ with $i > j$. Note that then $\mathfrak{A}(N_-)_s$ has basis consisting of

$$\{F_\sigma \big| \sigma \text{ is lower-triangular of weight} \leq s\}$$

Lemma 4.3.1. *Let σ be a lower-triangular N-shift. Let $1 \leq k < l \leq N$.Then we may write*

$$E_{l,k} F_\sigma = F_{E_{l,k}+\sigma} + \sum_{p=1}^{L} c_p \cdot F_{\sigma_p}$$

where L is a non-negative integer (which may be zero), each c_p is a complex number, and

$$\sigma_1, \cdots, \sigma_L$$

are lower-triangular N-shifts of weight smaller than that of σ.

PROOF:
Immediate.

Lemma 4.3.2. *Let σ be a lower-triangular N-shift. Express the Weyl polarization $P(\sigma)$ as a $\mathbb{C}$-linear combination of the basis $\mathfrak{B}(<<)$ for $\mathfrak{A}(N_-)$, say:*

$$P(\sigma) = \sum \{C_\tau \cdot F_\tau \big| \tau \in (\Pi^N)^{LT}\} \tag{4.3.2}$$

(where all C_τ are complex numbers). Then

$$C_\sigma = \frac{1}{(\sigma)!}\,,$$

and

$$C_\tau \neq 0 \Rightarrow W(\tau) < W(\sigma)\,.$$

PROOF:

We argue by induction on $W = W(\sigma)$:

If W=1, then for suitable k, l in $\underline{N}$, we have

$$k < l, \sigma = E_{l,k}\,.$$

Hence $P(\sigma) = D_{l,k} = F_{E_{l,k}}$, $\sigma! = 1$, and the assertion is clear in this case.

Next, assume that $W > 1$, and that the Lemma to be proved holds for all lower-triangular N-shifts of weight $< W$.

It follows immediately from the induction hypothesis, that if τ is a lower-triangular N-shift of weight $W' < W$, then $P(\tau)$ lies in $\mathfrak{A}(N_-)_{W'}$.

There exist integers k, l with

$$1 \leq k < l \leq N,\ \sigma_{l,k} > 0\,.$$

By Cor.2.3.2, there then exist lower-triangular N-shifts

$$\sigma_1, \cdots, \sigma_L$$

(with $L \geq 0$), each of weight $W - 1$, and positive integers $m_1, \cdots, m_L$, such that

$$\sigma_{l,k} \cdot P(\sigma) = E_{l,k} P(\sigma - E_{i,j}) - \sum_{s=1}^{L} m_s P(\sigma_s)\,. \tag{4.3.3}$$

By the induction hypothesis applied to the N-shift $\sigma - E_{l,k}$ of weight $W-1$, we may write

$$P(\sigma - E_{l,k}) = \frac{1}{(\sigma - E_{l,k})!} F_{\sigma - E_{l,k}} + (\text{ an element in } \mathfrak{A}(N_-)_{W-2})\,.$$

Hence, using Lemma 4.3.1, it follows that

$$E_{l,k} P(\sigma - E_{l,k}) = \frac{1}{(\sigma - E_{l,k})!} F_\sigma + (\text{ an element in } \mathfrak{A}(N_-)_{W-1})\,.$$

Since

$$(\sigma - E_{l,k})! = \frac{1}{\sigma_{l,k}} \cdot (\sigma)!\,,$$

we may rewrite (4.3.3) as

$$P(\sigma) = \frac{1}{\sigma!} F_\sigma + (\text{ an element in } \mathfrak{A}(N_-)_{W-1})$$

which completes the induction, and so the proof of Lemma 4.3.2.

Our present purpose is the proof of assertion A3, i.e., the proof that γ satisfies the Shapovalov normalization property VS2) explained in §3.1. Let σ_0 denote the N-shift

$$\sigma_0 = r(E_{i+1,i} + \cdots + E_{j-1,j})\,. \tag{4.3.4}$$

The assertion to be proved is, that when γ is expressed as a $\mathbb{C}$-linear combination of the basis (4.3.1) for $\mathfrak{A}(N_-)$, the coefficient of the basis vector F_{σ_0} is precisely 1.

To see this, let us recall eqn.(4.1), which defined γ as:

$$\gamma \stackrel{\text{def}}{=} \sum_{\sigma \in \mathcal{T}(\tau)} \langle \sigma ; \tau \rangle P(\sigma) \tag{4.3.5}$$

where

$$\mathcal{T}(\tau) = TERM(i,j,r)\,,$$

is the set of lower-triangular N-shifts subordinate to $(\lambda_i - \lambda_j, r)$, as furnished by Def.3.1.1. In particular, the three conditions of Def.3.1.1 are satisfied by the N-shift (4.3.4), i.e. $\mathcal{T}(\tau)$ contains σ_0.

CLAIM: σ_0 has strictly larger weight than any other member of $\mathcal{T}(\tau)$.

PROOF: If σ_1 is any element of $\mathcal{T}(\tau)$, and σ_1 is distinct from σ_0, then we may write

$$\sigma_1 = E_{l,k} + \sigma_1'$$

with

$$i \le k < l \le j,\ l - k > 1, \text{ and } \sigma_1' \text{ effective.}$$

Then the N-shift

$$E_{l,k+1} + E_{k+1,k} + \sigma_1'$$

lies in $\mathcal{T}(\tau)$, and has weight larger by 1 than that of σ_1—which completes the proof of the Claim.

Combining the result just proved with Lemma 4.3.2 and eqn.(4.3.5), we obtain:

$$\gamma = \frac{\langle \sigma_0 ; \tau \rangle}{\sigma_0 !} F_{\sigma_0} + (\text{ an element in } \mathfrak{A}(N_-)_{r(i-j)-1})$$

i.e., there exist distinct lower-triangular

$$\sigma_1, \sigma_2, \cdots, \sigma_L ,$$

all of weight strictly less than that of σ_0, and complex numbers C_s, such that

$$\gamma = \frac{\langle \sigma_0 ; \tau \rangle}{\sigma_0 !} F_{\sigma_0} + \sum_{s=1}^{L} C_s F_{\sigma_s} \tag{4.3.6}$$

Let us now appeal to Def.3.2.2 in order to compute $\langle \sigma_0 ; \tau \rangle$. Here we must replace each $K_\sigma(k)$ in eqn.(3.2.2) by r, thus obtaining

$$\langle \sigma_0 ; \tau \rangle = (r!) \cdot \prod_{k=i+1}^{j-1} [r!0! \binom{l_i - l_k}{0}] = (r!)^{j-i}$$

But clearly eqn.(4.3.4) implies that

$$(\sigma_0)! = (r!)^{j-1}$$

Hence, we see that the coefficient of F_{σ_0} in the right-hand side of eqn.(4.3.6) is 1, as was to be proved.

This completes the proof of Assertion A3, hence of Theorem 3.3.4, and so also of all the assertions in §3.

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